Second-order moments of frequency asymmetric cycles

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Second-order moments of frequency asymmetric cycles[*]

Miguel Artiach [**]

Abstract

Second-order moments, as even functions in time, are conventionally regarded as containing no information about the time irreversible nature of a sequence and therefore about its frequency asymmetry. However, this paper shows that the frequency asymmetry produces a clearly distinct behaviour in second-order moments that can be observed in both the time domain and the frequency domain. In addition, a frequency domain method of estimation of the differing lengths of the recessionary and expansionary stages of a cycle is proposed and its finite sample performance evaluated. Finally, the asymmetric patterns in the waves of the US unemployment rate and in the sunspot index are analysed.

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1 Introduction

There is a great ongoing controversy whether or not the business cycle has recessions and expansions with differing lengths, a stylised fact that was firstly suggested by Mitchell (1927), Keynes (1936) and Hicks (1950). This phenomenon, known as cyclical frequency or steepness asymmetry, raises a relevant econometric issue as it implies the time irreversibility of the series[1]. All stationary Gaussian processes are time reversible (Weiss, 1975); therefore conventional linear Gaussian forecasting techniques are inappropriate to capture the asymmetries of the business cycle. This has led to a broad set of proposals that attempt to detect and measure the frequency asymmetry from a quantitative perspective.

A first line of approach roots in the work by Brillinger and Rosenblatt (1967) who show that a strictly stationary time series is time reversible if and only if the imaginary parts of all the higher-order spectra are identically zero. Hinich and Rothman (1998) exploit this characteristic and propose a test based on the bispectrum. They find evidence of time irreversibility in the industrial production index, the consumer price index and the unemployment rate of various OECD countries. Lim et al. (2008), in a different context, develop a test based on the next polyspectrum, the trispectrum, and report time irreversibility patterns on financial returns of daily series of 48 stock markets worldwide. Higher-order moments have also been used from a time-domain point of view. In this sense, Ramsey and Rothman (1996) report that the paired third-order generalized autocovariances of a broad set of US macroeconomic series show evidence of asymmetry.

Other approaches focus on the first differences of the series. Neftci (1984) uses the signs of the first differences as indicator sequences for periods of expansion and contraction. He derives a test for symmetry in terms of transition probabilities assuming that the indicator sequences are generated by second-order Markov processes and reports evidence of longer periods of expansion in the US unemployment rate. Using this approach, Hamilton (1989) reports the presence of asymmetric patterns in the US GDP. Rothman (2008) confirms the existence of Neftci-type asymmetries in an updated US unemployment rate series. Sichel (1993) uses the sample skewness of the first difference and also reports asymmetries in the unemployment.

Different non-linear models have been proposed to capture a cyclical asymmetric behaviour. Teräsvirta and Anderson (1992) fit *smooth transition* autoregressive models to thirteen international production series. The dy-

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[1]A series x_t is time irreversible (or directional) if for every positive integer n, and every $t_1, t_2, \ldots, t_n \in \mathbb{R}$, the vectors $\{x(t_1), x(t_2), \ldots, x(t_n)\}$ and $\{x(-t_1), x(-t_2), \ldots, x(-t_n)\}$ do not possess the same joint probability distributions (Brillinger and Rosenblatt, 1967)

namics of these STAR models indicate a difference between *low* (recessionary) *regimes* and *upper* (expansionary) *regimes*. Brännäs and De Gooijer (1994) introduce an autoregressive-asymmetric moving average model that captures asymmetry via a different specification of the MA polynomial for expansions and recessions and derives a Wald test that rejects the null of symmetry in the US real GNP growth rate. Rothman (1998) reports that the use of non-linear time series models for the U.S. unemployment rate improves the forecast performance. Crespo (2003) proposes an asymmetric cyclical unobserved components (UC) model with a stochastic sine-cosine wave with a regime-dependent frequency and develops a test for symmetry which reveals asymmetries in the US rates of unemployment and industrial production.

By contrast, some papers fail to find evidence of cyclical asymmetries over the business cycle. Using Neftci's procedure, Falk (1986) and Westlund and Ohlen (1991) report no sustainable signs of asymmetry in a large number of international macroeconomic time series. More recently, Peiro (2004), using conventional t and F techniques, tests for the equality of the distributions of contractions and expansions in the quarterly industrial production indexes of France, Germany and the United Kingdom and does not reject the null of symmetry.

All the aforementioned works focus on the dynamics of series that alternate decreasing and increasing stages without an overall sense of periodicity. This approach is appropriate in Economics, where the oscillations of the activity are prone to not possess a fixed period, and permits to treat each macroeconomic series as a whole despite the variability in the length of the consecutive waves. In this paper, nevertheless, the frequency asymmetry is studied deterministically as behaviour confined within a pre-established period that is split between both phases of expansion and contraction in an unbalanced way and the second order moments of such behaviour are analysed. Second order moments, as even functions in time, are conventionally regarded as containing no information about the time irreversible nature of data and therefore on the frequency asymmetry. However, in this work it is shown that the frequency asymmetry produces clearly distinct behaviour of second order moments that can be observed in both the time domain and the frequency domain.

The major contributions of the current work are threefold. Firstly, this proposal provides a new way to perceive a frequency asymmetric behaviour through the two most widely used graphical tools in time series analysis: the correlogram and the periodogram, so assessment of asymmetry can be made by simple visual inspection of one or both of these graphics. The second main contribution is a new estimation procedure of the differing frequencies of expansion and contraction. This approach, based on the Fourier represen-

tation of frequency asymmetric cycles, implements a non-linear least squares method to produce simultaneous estimates of the asymmetric frequencies, the phase and the amplitude of a deterministic asymmetric cycle. Finally, the expansion and contraction lengths in every wave of the US unemployment rate are estimated and further evidence of asymmetry in the business cycle is reported. It is shown that the frequency asymmetry is a prominent feature of most but not all the fluctuations of the business cycle.

The structure of the paper is as follows: Section 2 defines the deterministic frequency asymmetric cycle and characterizes its mean and variance. Sections 3 and 4 derive respectively the Autocovariance Function and the Periodogram of such a cycle and show how these statistics vary with the degree of asymmetry. Section 5 introduces the Asymmetric Cyclical Regression Model and proposes a non-linear least squares (NLLS) procedure of estimation of its parameters and Section 6 studies the finite sample performance of the procedure for a wide range of parameter values, sample lengths and variances of the error term. Two empirical applications are analysed in Section 7, the first one focuses on the frequency asymmetries of the business cycle as expressed in the seasonally adjusted US Unemployment Rate and the second one explores the dynamics of the sunspot index. Finally, Section 8 concludes. The proofs of the main propositions are relegated to the appendix.

2　The deterministic frequency asymmetric cycle

Consider the following deterministic cycle completed in $\tau \geq 3$ observations

$$x_t = \begin{cases} \rho \cos\left[\lambda^-\left(t + \varphi\right)\right] & \text{if } t \leq a^- - \varphi \\ -\rho \cos\left[\lambda^+\left(t - a^- + \varphi\right)\right] & \text{if } a^- - \varphi < t \leq \tau - \varphi \\ \rho \cos\left[\lambda^-\left(t - \tau + \varphi\right)\right] & \text{if } \tau - \varphi < t \leq \tau \end{cases} \qquad (1)$$

where $t = 1, 2, \ldots, \tau$, ρ is the amplitude, $\{\lambda^-, \lambda^+\} \in \left[\frac{\pi}{\tau-1}, \pi\right]$ are the frequencies that determine respectively the velocity of contraction and the velocity of expansion of the cycle, $a^- = \frac{\pi}{\lambda^-}$ is the half-period of contraction and $\varphi \in [0, \tau)$ is the phase shift that settles the starting point of the cycle. $\{\tau, \varphi, a^-\} \in \mathbb{N}$ is assumed with a small loss of generality[2]. When $\lambda^+ = \lambda^-$,

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[2]The asymmetric cycle presented in (1) is a discretization of an equivalently defined continuous function with $t \in \mathbb{R}^+$. Then, when $\{\tau, \varphi, a^-\} \in \mathbb{N}$ each half-period of the cycle is evaluated at equispaced points fully covering the interval $[-\rho, \rho]$ and the results presented henceforth are exact in this case. They can be extended asymptotically as $\tau \to \infty$ and/or $T \to \infty$, where T is the length of a series containing $k > 1$ concatenations of x_t.
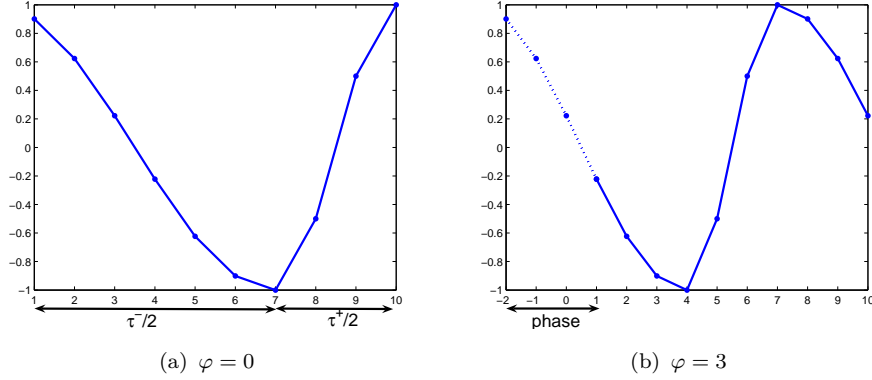
(a) $\varphi = 0$　　　　　　　　　(b) $\varphi = 3$

Figure 1: Asymmetric cycle: $\rho = 1$, $\tau = 10$, $\lambda^+ = \pi/3$ and $\lambda^- = \pi/7$.

(1) represents the standard symmetric cycle. If $\lambda^+ > \lambda^-$ the cycle grows more quickly than it decreases, taking place the opposite effect for $\lambda^+ < \lambda^-$ (thus, for example in unemployment series $\lambda^+ > \lambda^-$ can be expected). The restriction $\tau \geq 3$ is introduced just as a precondition for asymmetry.

Figure 1 shows two realizations of an asymmetric cycle with amplitude $\rho = 1$, period $\tau = 10$, $\lambda^+ = \pi/3$ and $\lambda^- = \pi/7$, zero phase in the first case and $\varphi = 3$ in the second. In Figure 1(a) an asymmetric cosinoidal wave that takes 7 steps to decrease and 3 to rise back can be observed. In this cycle both half-periods occur consecutively and unbrokenly. On the contrary, the cycle in Figure 1(b) starts at a midpoint of the contraction half-period and then completes its three step expansion prior to decreasing back to the starting point due to a phase shift of 3 lags.

Proposition 1 characterizes the mean and the variance of the deterministic asymmetric cyclical sequence defined in (1). The following notation will be frequently used henceforth: $a^+ = \frac{\pi}{\lambda^+} = \frac{\tau^+}{2}$ and $a^- = \frac{\pi}{\lambda^-} = \frac{\tau^-}{2}$ are the half-periods of expansion and contraction of the cycle (the latter was already defined), then let $a_{min} = \min(a^+, a^-)$, $a_{max} = \max(a^+, a^-)$, $\lambda_{min} = \frac{\pi}{a_{max}} = \min(\lambda^+, \lambda^-)$ and $\lambda_{max} = \frac{\pi}{a_{min}} = \max(\lambda^+, \lambda^-)$.

Proposition 1: *Let x_t be as defined in (1). Then the mean and variance of x_t are:*

- $M_x = 0$

- $S_x^2 = \begin{cases} \dfrac{\rho^2}{2} & \text{if } \lambda_{max} \neq \pi \\ \dfrac{\rho^2}{2}\left(1 + \dfrac{1}{\tau}\right) & \text{if } \lambda_{max} = \pi \end{cases}$

Proposition 1 shows that the mean of asymmetric cycles is constant for all degrees of asymmetry and the variance is also constant except if the cycle covers all its stage of expansion or of contraction in one single step, which implies $\lambda_{max} = \pi$.

3 Autocovariance function of frequency asymmetric cycles

In the time domain, the cyclicity of a periodic sequence x_t appears in second-order moments as cosinoidal waves of equivalent period in the autocorrelation function (ACF), defined as

$$r(k) = \frac{s(k)}{s(0)}, \tag{2}$$

where $s(k)$ are the autocovariances calculated as

$$s(k) = \frac{1}{N} \sum_{t=1}^{N-k} (x_{t+k} - \bar{x})(x_t - \bar{x}).$$

Considering complete cycles with respect to every lag k, the autocovariances of (1) can be defined as

$$s(k) = \frac{1}{\tau} \sum_{t=1}^{\tau} x_{t+k} x_t \tag{3}$$

where $x_i = x_{i-\tau}$ for $i > \tau$.

Proposition 2 characterizes the autocovariance function of the deterministic asymmetric cycle x_t.

Proposition 2: *Let x_t satisfy (1), then the autocovariance function of x_t*

verifies

$$
s\left(k\right)=\begin{cases}
\dfrac{\rho^2}{\tau}\left\{\dfrac{a_{max}-k}{2}\cos\left(\lambda_{min}k\right)+\dfrac{a_{min}-k}{2}\cos\left(\lambda_{max}k\right)\right.\\
\quad-\dfrac{\cot\lambda_{max}\sin\left(\lambda_{max}k\right)}{2}-\dfrac{\cot\lambda_{min}\sin\left(\lambda_{min}k\right)}{2}\\
\quad\left.+\dfrac{\sin\left(\lambda_{max}k\right)\sin\lambda_{max}-\sin\left(\lambda_{min}k\right)\sin\lambda_{min}}{\cos\lambda_{min}-\cos\lambda_{max}}\right\}&\text{if }\lambda^-\neq\lambda^+,\ k\leq a_{min}\ \text{and}\ a_{min}>1\\[6pt]
\dfrac{\rho^2}{\tau}\left\{\dfrac{a_{max}-k}{2}\cos\left(\lambda_{min}k\right)-\dfrac{\cot\lambda_{min}\sin\left(\lambda_{min}k\right)}{2}\right.\\
\quad\left.-\sin\left(\lambda_{min}k\right)\tan\left(\dfrac{\lambda_{min}}{2}\right)+\left(1-k\right)\left(-1\right)^k\right\}&\text{if }\lambda^-\neq\lambda^+\ \text{and}\ k\leq a_{min}=1\\[6pt]
\dfrac{\rho^2}{\tau}\left\{\dfrac{a_{max}-k}{2}\cos\left(\lambda_{min}k\right)-\dfrac{k-a_{min}}{2}\cos\left[\lambda_{min}\left(k-a_{min}\right)\right]\right.\\
\quad-\dfrac{\cot\lambda_{min}\sin\left(\lambda_{min}k\right)}{2}-\dfrac{\cot\lambda_{min}\sin\left[\lambda_{min}\left(k-a_{min}\right)\right]}{2}\\
\quad\left.+\dfrac{2\sin\left[\lambda_{min}\left(k-\frac{a_{min}}{2}\right)\right]\cos\left(\frac{a_{min}\lambda_{min}}{2}\right)\sin\lambda_{min}}{\cos\lambda_{max}-\cos\lambda_{min}}\right\}&\text{if }\lambda^-\neq\lambda^+\ \text{and}\ a_{min}<k\leq\frac{\tau}{2}\\[6pt]
\dfrac{\rho^2\cos\left(\lambda^{asy}k\right)}{2}&\text{if }\lambda^-=\lambda^+
\end{cases}
$$

(4)

where $s\left(\tau-k\right)=s\left(k\right)$, $l\tau\leq k\leq l\tau+\frac{\tau}{2}$ and $l=0,1,2,\ldots$

In periodic symmetric series, the autocorrelation function is negative in the vicinity of $\tau/2$ and equals -1 if $k=\tau/2$ as a consequence of the periodic component passing maximally out of phase with its counterpart. If the series is asymmetric this situation does no longer hold and so the serial dependence at $\tau/2$ decreases in absolute value as the degree of asymmetry increases, a behaviour that is extensive to all autocovariances in the vicinity of $\tau/2$. The autocovariances located at multiples of the period, on the contrary, remain invariant to asymmetry (except for $\lambda_{max}=\pi$) as the series is still periodic and they are consequently equal to the variance of the series. Then, the autocorrelation at $k=\tau$ is always 1, independently of the degree of asymmetry, whilst it generally decreases in absolute value at $k\neq\tau$ as the degree of asymmetry increases. This produces the most visually identifiable feature of frequency asymmetry in the ACF that corresponds to a wave with less deep troughs than peaks are high, a characteristic that becomes more outstanding as $\lambda_{max}\to\pi$. The frequency asymmetry of the series therefore becomes amplitude asymmetry in the correlogram.

Figure 2 shows five realizations of two waves of cycles with the same period $\tau=10$ and an increasing degree of asymmetry: $\{\lambda^+=\lambda^-=\pi/5\}$, $\{\lambda^+=\pi/4,\ \lambda^-=\pi/6\}$, $\{\lambda^+=\pi/3,\ \lambda^-=\pi/7\}$, $\{\lambda^+=\pi/2,\ \lambda^-=\pi/8\}$ and $\{\lambda^+=\pi,\ \lambda^-=\pi/9\}$. They are accompanied by their respective correlograms and periodograms (which will be analysed in the next section) calculated according to (2) with (3) and (5).

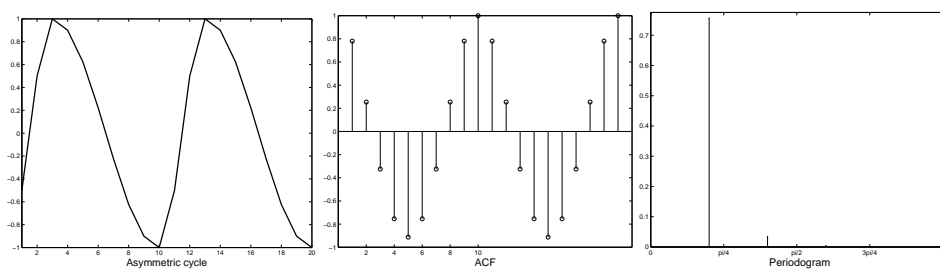In Figure 2(a), the wave of the ACF holds troughs and peaks of equal amplitude. In Figures 2(b) to 2(d) it can be seen how the increasing degree
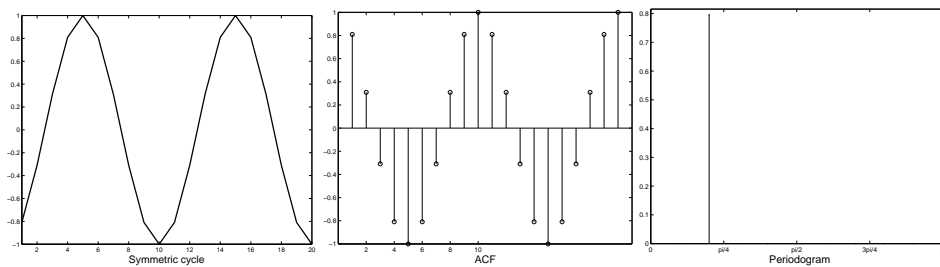
(a) $\lambda^+ = \lambda^- = \pi/5$
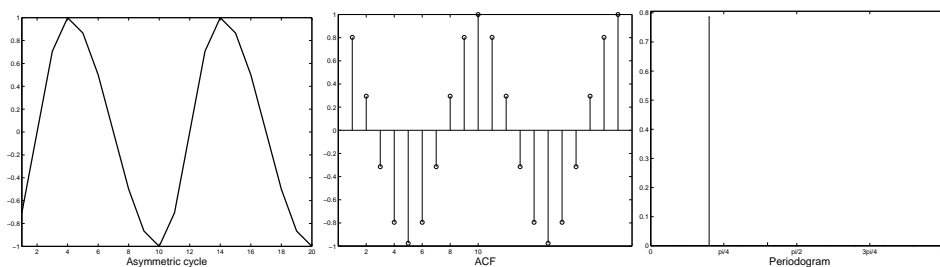
(b) $\lambda^+ = \pi/4$, $\lambda^- = \pi/6$

(c) $\lambda^+ = \pi/3$, $\lambda^- = \pi/7$
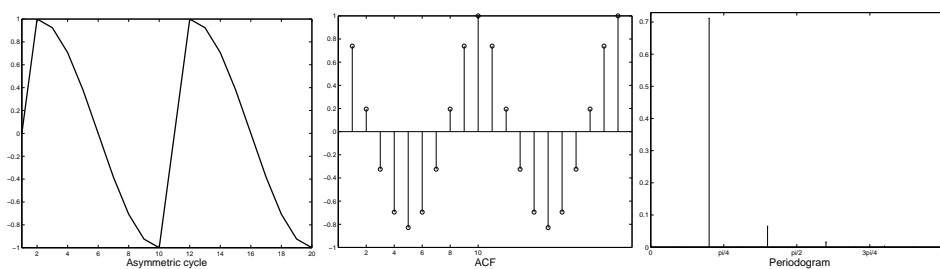
(d) $\lambda^+ = \pi/2$, $\lambda^- = \pi/8$

Figure 2: Cycles, correlograms and periodograms at increasing degrees of asymmetry

(e) $\lambda^+ = \pi$, $\lambda^- = \pi/9$
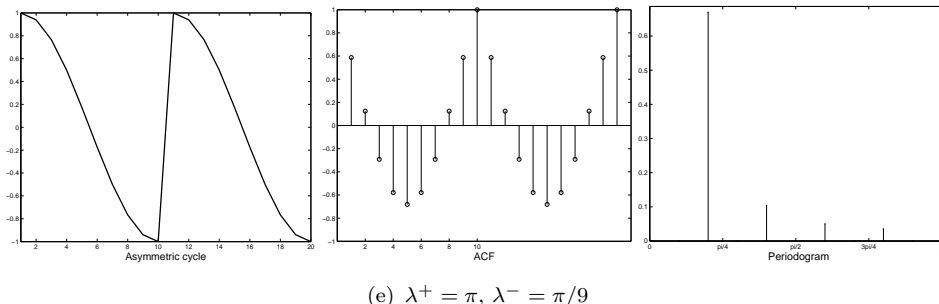
Figure 2: Cycles, correlograms and periodograms at increasing degrees of asymmetry (Cont.)

of asymmetry decreases all the autocorrelations except the ones located at multiple values of τ. This produces the visual effect of shrinking troughs and constant peaks. Finally, Figure 2(e) shows that under complete frequency asymmetry ($\lambda_{max} = \pi$) the amplitude of the troughs is clearly shorter than the amplitude of the peaks.

4 Periodogram of frequency asymmetric cycles

The frequency domain settles a natural framework for the analysis of the existence and characteristics of the cycles since they are revealed as sharp peaks or poles in the periodogram of x_t defined as

$$
\begin{aligned}
I_x\left(\lambda_j\right) &= \left| \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} x_t e^{-i\lambda_j t} \right|^2 \\
&= \frac{1}{2\pi T} \left[\left\{ \sum_{t=1}^{T} x_t \cos\left(\lambda_j t\right) \right\}^2 + \left\{ \sum_{t=1}^{T} x_t \sin\left(\lambda_j t\right) \right\}^2 \right].
\end{aligned}
\tag{5}
$$

Proposition 3 explores how the asymmetric behaviour of a cycle affects its periodogram. Firstly, Lemma 1 defines the Fourier frequencies of asymmetric cycles as functions of the frequencies of expansion and contraction.

Lemma 1: *Let x_t satisfy (1), then the Fourier frequencies of x_t are $\lambda_j = j\lambda^{asy}$ with $j = 1, \ldots, n$, where n is the integer part of $\tau/2$ and λ^{asy} is the frequency corresponding the period of x_t ($\lambda^{asy} = 2\pi/\tau$) that verifies*

$$
\lambda^{asy} = \frac{2\lambda^+ \lambda^-}{\lambda^+ + \lambda^-}.
\tag{6}
$$

PROOF : By definition $\tau = a^+ + a^- = \pi/\lambda^+ + \pi/\lambda^- = 2\pi/\lambda^{asy}$ and solving for λ^{asy} we get (6).

Proposition 3: *Let x_t satisfy (1). Then the periodogram of x_t verifies*

$$
I(\lambda_j) = \begin{cases}
\dfrac{\rho^2}{4\lambda^{asy}} & \text{if } \lambda^- = \lambda^+ \text{ and } j = 1 \\[2mm]
0 & \text{if } \lambda^- = \lambda^+ \text{ and } j > 1 \\[2mm]
\dfrac{\rho^2}{2\pi\tau} & \text{if } \lambda^- \neq \lambda^+ \text{ and } j = j^* = \frac{\tau}{2} \\[2mm]
\dfrac{\rho^2}{8} \dfrac{\lambda_{min}}{\lambda_{max}(\lambda^+ + \lambda^-)} & \text{if } \lambda^- \neq \lambda^+ \text{ and } j = j^* \neq \frac{\tau}{2} \\[2mm]
\dfrac{\rho^2 \lambda^{asy}}{4\pi^2} \left[\dfrac{\left(\cos\lambda^- - \cos\lambda^+\right)\cos\left(\frac{\pi j}{2j^*}\right)\sin\lambda_j}{(\cos\lambda_j - \cos\lambda^-)(\cos\lambda_j - \cos\lambda^+)} \right]^2 & \text{if } \lambda^- \neq \lambda^+ \text{ and } j \neq j^*
\end{cases}
\tag{7}
$$

where $j^* = \dfrac{\lambda^+ + \lambda^-}{2\lambda_{min}}$.

Proposition 3 shows that the periodogram ordinates of asymmetric cycles are not zero at the multiples of λ^{asy}, whereas only λ^{asy} possesses spectral power in the symmetric case.

However, two exceptions must be considered. The first one arises as

$$
I(\lambda_j) = 0 \text{ at } j = (2k-1)j^*, \text{ for } k \geq 2,
\tag{8}
$$

and establishes that for j^* an integer the periodogram of x_t is zero at odd multiples of j^* -$3j^*$, $5j^*$, ...- whereas it is some positive value at any other frequency including j^* and even multiples of j^*. As a result the periodogram is alternating zero and positive values at frequencies that are multiples of λ_{max}, starting at $3\lambda_{max}$.

The second exception takes place at

$$
I(\pi) = 0 \text{ except for } \lambda^+ \neq \lambda^- \text{ and } \lambda_{max} = \pi,
\tag{9}
$$

and states that the periodogram is always zero at $\lambda_j = \pi$, specifically due to the $\sin(\lambda_j)$ term in the asymmetric case, except for $j = j^* = \frac{\tau}{2}$.

Taking into account Parseval's Theorem (see Priestley, 1981, Eq. 4.5.12), Propositions 1 and 3 imply

$$
I_{\lambda^+ = \lambda^-}\left(\lambda^{asy}\right) \leq \sum_{j=1}^{n} I_{\lambda^+ \neq \lambda^-}\left(\lambda_j\right),
$$

where $I_{\lambda^+ = \lambda^-}$ and $I_{\lambda^+ \neq \lambda^-}$ are the periodograms of symmetric and asymmetric cycles respectively and the inequality holds only if $\lambda_{max} = \pi$. Then the

variance of an asymmetric cycle distributes throughout all Fourier frequencies -except if (8) or (9) hold. As a consequence, the periodogram value at λ^{asy} decreases as the degree of asymmetry increases and reaches its minimum when $\lambda_{max} = \pi$. This can lead to low power in tests for deterministic cycles that are based on the maximum of the periodogram, such as the widely used g-test of Fisher (1929).

The periodograms depicted in Figure 2 exemplify how this graphic evolves with the degree of asymmetry of the cycle. In Figure 2(a) the cyclical symmetry produces only one non-zero ordinate at $\lambda^{asy} = \pi/5$. The periodograms in Figures 2(b) to 2(d) show how an increasing portion of the variability of the series is distributed along the multiples of $\pi/5$ as the distance between the asymmetric frequencies enlarges.

These figures also allow to appreciate that $I(\lambda_{j-1}) > I(\lambda_j)$, a feature that verifies for $j < 3j^*$, that is, the periodogram decreases monotonically up to the first frequency where it could turn zero -as stated in (8)- which implies at least the first four Fourier frequencies ($j = 3j^* \Rightarrow j > 3$) if they are defined. This characteristic becomes more evident as $\lambda_{max} \to \pi$, as in this limit $j = 3j^*$ is never reached, and provides a visual criterion to distinguish the frequency asymmetry from other periodic behaviour that can display peaks of any size at these frequencies, which includes a seasonal pattern if the period length corresponds to one year, or a symmetric cycle, that has zero-ordinates except for λ^{asy}.

Finally, in the last cycle of complete asymmetry (Figure 2(e)) the periodogram value at λ^{asy} reaches its minimum, as it has been said, whereas the corresponding values at $j\lambda^{asy}$, $j > 1$, reach their maxima. The exception posited in (9) is clearly perceived in the non-zero periodogram ordinate that appears at $\lambda_j = \pi$ in this last figure.

5 NLLS Estimation of λ^+, λ^-, φ and ρ

The discrete version of Fourier's Theorem states that any discrete function evaluated at finite points can be approximated up to any desired degree by a finite sum of paired cosine and sine terms. In the simplest case, if the series is actually a symmetric cycle, its Fourier representation is

$$x_t = \alpha \cos(\lambda t) + \beta \sin(\lambda t) \tag{10}$$

where the parameters α and β characterize both the amplitude and the phase of the cycle as $\rho^2 = \alpha^2 + \beta^2$ and $\varphi = -\frac{\arctan(\beta/\alpha)}{\lambda}$ respectively. The periodogram ordinate at λ is then proportional to ρ^2 and, as it has already been pointed out, is zero in the rest of frequencies.

The result in (7), on the contrary, shows that the periodogram of an asymmetric cycle has non-zero ordinates at the multiples of the frequency of the period (with the exceptions of (8) or (9) when applicable). Therefore the Fourier representation of such cycles requires the sum of n pairs of cosine and sine terms, as described in the next proposition.

Proposition 4: (Fourier series of frequency asymmetric cycles) *Let x_t satisfy (1) with $\lambda^+ \neq \lambda^-$. Then the Fourier representation of x_t is*

$$x_t = \sum_{j=1}^{n-1} \alpha_j \cos(\lambda_j t) + \beta_j \sin(\lambda_j t) + \alpha_n \tag{11}$$

where the Fourier coefficients α_j and β_j depend on the amplitude ρ, on the phase φ and on the asymmetric frequencies λ^+ and λ^- and are defined as

$$\alpha_j = \frac{2\rho}{\tau} \begin{cases} \frac{a_{min} \cos(\lambda_{max}\varphi)}{2} & \text{if } j = j^* \\ \frac{\left(\cos\lambda^- - \cos\lambda^+\right) \sin\lambda_j \cos\left(\frac{a^-\lambda_j}{2}\right) \sin\left(\frac{a^-\lambda_j}{2} - \lambda_j\varphi\right)}{(\cos\lambda_j - \cos\lambda^-)(\cos\lambda_j - \cos\lambda^+)} & \text{if } j \neq j^*, \end{cases}$$

and

$$\beta_j = -\frac{2\rho}{\tau} \begin{cases} \frac{a_{min} \sin(\lambda_{max}\varphi)}{2} & \text{if } j = j^* \\ \frac{\left(\cos\lambda^- - \cos\lambda^+\right) \sin\lambda_j \cos\left(\frac{a^-\lambda_j}{2}\right) \cos\left(\frac{a^-\lambda_j}{2} - \lambda_j\varphi\right)}{(\cos\lambda_j - \cos\lambda^-)(\cos\lambda_j - \cos\lambda^+)} & \text{if } j \neq j^*. \end{cases}$$

and

$$\alpha_n = \begin{cases} \frac{\rho\cos(\pi\varphi)}{\tau} & \text{if } \lambda_{max} = \pi \\ 0 & \text{otherwise.} \end{cases}$$

If τ is odd the last term in (11) does not appear and the summation runs for $j = 1, \dots, n$.

The proof is omitted as it is straightforward taking into account the Fourier representation of a time series (see Harvey, 1993, page 60) and the proof of Proposition 3.

Note that for $a^+ \in (1, \tau - 1) - \left\{\frac{\tau}{2}\right\}$ a simpler formula for (11) is

$$x_t = \sum_{j=1}^{\left[\frac{\tau-1}{2}\right]} \delta_j(t), \tag{12}$$

where $[\cdot]$ means "the integer part of" and

$$\delta_j(t) = \frac{2\rho}{\tau} \begin{cases} \frac{a_{min} \cos(\lambda_{max}\varphi + \lambda_j t)}{2} & \text{if } j = j^* \\ \frac{\left(\cos\lambda^- - \cos\lambda^+\right) \sin\lambda_j \cos\left(\frac{a^-\lambda_j}{2}\right) \sin\left[\lambda_j\left(\frac{a^-}{2} - \varphi - t\right)\right]}{(\cos\lambda_j - \cos\lambda^-)(\cos\lambda_j - \cos\lambda^+)} & \text{if } j \neq j^*. \end{cases}$$

Eq. (12) corresponds to the Fourier series of a pure deterministic cycle as defined in (1). This specification can be used to introduce the Asymmetric Cyclical Regression Model (ACRM) defined as

$$y_t = x_t + \varepsilon_t \tag{13}$$

where x_t is as in (12) and ε_t is a $iid(0, \sigma_\varepsilon^2)$ process.

The Signal-to-Noise Ratio (SNR) of the ACRM is defined as

$$SNR = \frac{\rho^2}{2\sigma_\varepsilon^2}. \tag{14}$$

The estimation of the parameters ρ, φ and one of the asymmetric frequencies λ^+ or λ^- (or equivalently one of the half-periods a^+ or a^-) of the ACRM can be achieved by non-linear least squares (NLLS) under the assumption of a known τ. Specifically, the NLLS estimator of $\boldsymbol{\Upsilon} = \{\rho, a^+, \varphi\}$ is defined as

$$\hat{\boldsymbol{\Upsilon}} = \arg \min_{P,A^+,\Phi} \sum_{t=1}^{T} \{y_t - x_t\}^2 \tag{15}$$

where x_t is as defined in (12) with $a^- = \tau - a^+$. A mean other than zero in y_t would not affect the results of this estimation strategy due to the orthogonality condition among all the regressors.

NLLS estimation strategies for the so-called *harmonic retrieval* (simultaneous estimation of the amplitude, the phase and also the frequency of a sinusoidal signal) have been studied by Walker (1971, 1973, 2003) and Hannan (1973) in symmetric cycles. They obtain consistency and asymptotic normality for a broad set of conditions on ε_t.

The asymptotic properties of (15) have yet to be established. For inferential purposes in this work the following distribution, based on the usual asymptotic properties of OLS procedures and the best linear approximation of (12) around the parameters, will be used:

$$\hat{\boldsymbol{\Upsilon}} \sim N \left[\boldsymbol{\Upsilon}, \sigma_\varepsilon^2 \left[\left(\frac{\partial \mathbf{x}\left(\hat{\boldsymbol{\Upsilon}}\right)}{\partial \boldsymbol{\Upsilon}} \right)' \left(\frac{\partial \mathbf{x}\left(\hat{\boldsymbol{\Upsilon}}\right)}{\partial \boldsymbol{\Upsilon}} \right) \right]^{-1} \right] \tag{16}$$

where $\mathbf{x}\left(\boldsymbol{\Upsilon}\right)$ is a $T \times 1$ vector containing the Fourier representation of the series defined in (12) for $t = 1, \ldots, T$.

$\sigma_\varepsilon^2 = 1$ $\sigma_\varepsilon^2 = 2.5$

(a) $SNR = 8$ (b) $SNR = 3.2$

(c) $SNR = 0.5$ (d) $SNR = 0.2$
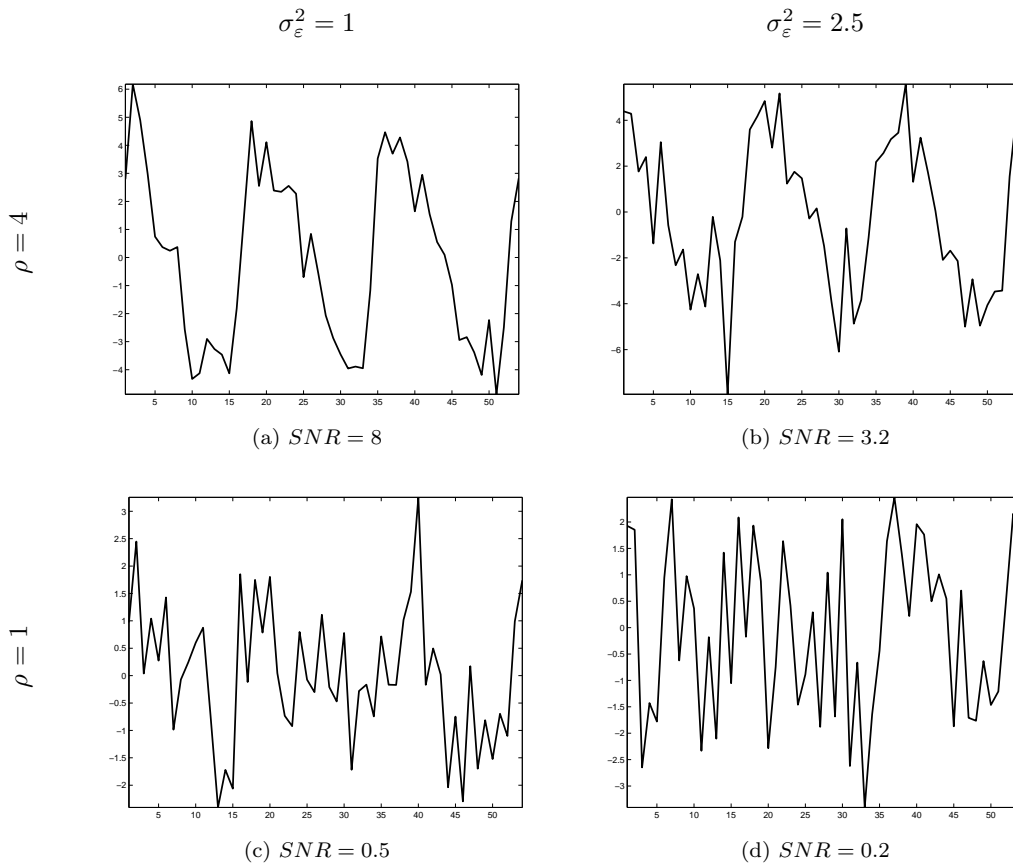
Figure 3: Asymmetric cycle with noise: $\lambda^+ = \pi/3, \lambda^- = \pi/15$.

6 Finite sample performance

In order to assess the finite sample performance of the NLLS estimation method proposed in the previous section a number of different combinations of period, amplitude, degree of asymmetry, phase and variance of the noise term were explored.

Two different amplitudes $\rho = 4, 1$ and variances of the noise term $\sigma_\varepsilon^2 = 1, 2.5$ were used, so four different SNR result from their combination, in decreasing order, $SNR = 8, 3.2, 0.5, 0.2$. The first and the last cases constitute examples of extremely high and extremely low SNR that allow to perceive more clearly the effect of this magnitude over the performance of the estimation procedure. The distribution of the noise is Gaussian $N(0, \sigma_\varepsilon^2)$ in all cases. Figure 3 depicts four samples of three waves of cycles with $\tau = 18$, $\lambda^- = \pi/15$ and $\lambda^+ = \pi/3$ and the four different $SNRs$ that exemplify the differing degrees of distortion faced by the estimation procedure.

Two different periods were used, $\tau = 18, 90$, and three different sample sizes $T = 180, 900, 2700$ for both periods so as a result each series contains 10, 50 or 150 waves of the cycle if $\tau = 18$ and 2, 10 and 30 if $\tau = 90$.

Different degrees of asymmetry and phases were imposed to the samples, specifically $a^+ = \left\{1, \frac{\tau}{6}, \frac{\tau}{3}, \frac{\tau}{2}, \frac{2\tau}{3}, \frac{5\tau}{6}, \tau - 1\right\}$ and $\varphi = \left\{0, \frac{\tau}{6}, \frac{\tau}{3}, \frac{\tau}{2}, \frac{2\tau}{3}, \frac{5\tau}{6}\right\}$. The cases $a^+ = \left\{1, \frac{\tau}{2}, \tau - 1\right\}$ are included as natural points of interest of our parametric space although the described procedure is not strictly applicable in these cases. No mean value was added.

1000 replications were generated in every combination of ρ, σ_ε^2, τ, T, a^+ and φ, summing up a total amount of 1008000 samples. The method was implemented using the *lsqnonlin* routine in Matlab 7.10.0 (Mathworks). An initial estimate of the amplitude parameter was calculated via the explained variance of the Fixed Frequency Effects Regression model $y_t = \sum_{j=1}^n \gamma_j \cos(\lambda_j t) + \kappa_j \sin(\lambda_j t) + \varepsilon_t$. Then $\hat{\rho}_0 = \sigma_{\hat{y}}\sqrt{2}$ where $\hat{y}_t = \sum_{j=1}^n \hat{\gamma}_j \cos(\lambda_j t) + \hat{\kappa}_j \sin(\lambda_j t)$ and $\hat{\gamma}_j$ and $\hat{\kappa}_j$ were estimated by OLS. The initial estimates of a^+ and φ were obtained from a prior exploration of (15) for a grid of values on the parametric space $A^+ \times \Phi = [1, \tau - 1] \times [0, \tau)$.

The following indicators for the estimation of a^+ were calculated in the 1000 replications of every parametric combination. The corresponding expressions for φ and ρ are similarly defined:

$$\bar{\hat{a}}^+ = \frac{1}{1000} \sum_{rep=1}^{1000} \hat{a}_{rep}^+,$$

$$SE_{\hat{a}^+} = \sqrt{\frac{1}{1000} \sum_{rep=1}^{1000} \left[\hat{a}_{rep}^+ - \bar{\hat{a}}^+\right]^2}$$

and

$$bias_{\hat{a}^+} = \bar{\hat{a}}^+ - a^+,$$

where SE accounts for Standard Error.

Tables 1 and 2 show the biases and SEs of estimation of the parameters a^+ and φ respectively. The statistics in these two first tables are presented as a function of a^+ and therefore are calculated over 6000 replications as no dependence on the value of φ was appreciated in the estimations. To this end the estimates of φ are expressed as deviations from the true parametric value ($\hat{\varphi} - \varphi$). In the case of ρ (Table 3) no dependence on a^+ was found either and so the presented results correspond to 42000 replications.

Table 1 displays the biases and standard errors of estimation of a^+. It can be seen that the bias of estimation is very low even for the lowest SNR and shortest sample length ($\rho = 1$, $\sigma_\varepsilon^2 = 2.5$, $T = 180$) as it reaches a maximum

		$\tau = 18$				$\tau = 90$			
		$\rho = 1$		$\rho = 4$		$\rho = 1$		$\rho = 4$	
T	a^+	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$
180	1	0.2361	0.7419	0.0058	0.0191	1.4933	4.4613	0.0380	0.1221
		(0.7147)	(1.9542)	(0.0342)	(0.0795)	(4.1729)	(10.9129)	(0.1353)	(0.4557)
	$\frac{\tau}{6}$	0.0166	0.4028	-0.0085	-0.0188	-0.0850	0.9372	-0.0187	0.0464
		(1.4988)	(2.8085)	(0.3347)	(0.5327)	(8.1821)	(14.9025)	(1.5898)	(2.6266)
	$\frac{\tau}{3}$	-0.0409	0.0792	0.0022	0.0106	-0.3438	-0.6625	0.0154	0.0828
		(1.8823)	(3.3438)	(0.3971)	(0.6281)	(9.5462)	(17.8230)	(1.9970)	(3.1980)
	$\frac{\tau}{2}$	-0.0109	0.0108	0.0033	0.0084	0.1915	-0.0534	-0.0032	-0.0190
		(1.9413)	(3.6432)	(0.4175)	(0.6700)	(10.0287)	(19.2486)	(2.0980)	(3.2728)
	$\frac{2\tau}{3}$	0.0275	-0.0462	-0.0001	-0.0074	0.4291	0.2463	-0.0047	0.0689
		(1.8637)	(3.3666)	(0.3951)	(0.6522)	(9.4877)	(18.4173)	(1.9949)	(3.2597)
	$\frac{5\tau}{6}$	-0.0260	-0.4181	0.0011	0.0071	0.3039	-1.6056	-0.0143	0.0765
		(1.4652)	(2.8331)	(0.3458)	(0.5429)	(7.9595)	(15.0444)	(1.6058)	(2.5876)
	$\tau - 1$	-0.2233	-0.7345	-0.0069	-0.0187	-3.0374	-7.1636	-0.0403	-0.2004
		(0.6431)	(1.9615)	(0.0345)	(0.0811)	(5.2726)	(12.7029)	(0.1467)	(0.7207)
900	1	0.0258	0.1007	0.0008	0.0028	0.1590	0.5523	0.0042	0.0141
		(0.1036)	(0.3241)	(0.0121)	(0.0208)	(0.5770)	(1.8160)	(0.0168)	(0.0523)
	$\frac{\tau}{6}$	-0.0218	0.0091	0.0014	-0.0014	-0.0223	-0.0857	-0.0017	-0.0055
		(0.6064)	(1.0139)	(0.1481)	(0.2349)	(2.9834)	(5.2293)	(0.7035)	(1.1151)
	$\frac{\tau}{3}$	-0.0077	0.0045	-0.0047	0.0039	-0.0169	0.0093	-0.0043	-0.0367
		(0.7254)	(1.2061)	(0.1803)	(0.2794)	(3.6117)	(6.0381)	(0.8779)	(1.3697)
	$\frac{\tau}{2}$	0.0083	0.0197	0.0014	-0.0075	0.0040	0.0507	0.0150	-0.0159
		(0.7695)	(1.2550)	(0.1867)	(0.3013)	(3.8170)	(6.4520)	(0.9227)	(1.4786)
	$\frac{2\tau}{3}$	0.0034	-0.0269	-0.0015	0.0135	0.0505	0.0578	0.0063	0.0135
		(0.7192)	(1.2052)	(0.1790)	(0.2837)	(3.6455)	(6.1138)	(0.8776)	(1.3781)
	$\frac{5\tau}{6}$	0.0268	0.0120	-0.0009	0.0025	0.0425	0.1845	0.0224	0.0234
		(0.6134)	(1.0057)	(0.1500)	(0.2403)	(2.9853)	(5.3004)	(0.6967)	(1.1101)
	$\tau - 1$	-0.0249	-0.0912	-0.0011	-0.0031	-0.2903	-1.3271	-0.0037	-0.0153
		(0.1028)	(0.2973)	(0.0122)	(0.0213)	(1.0066)	(2.7754)	(0.0164)	(0.0603)
2700	1	0.0062	0.0216	0.0003	0.0008	0.0388	0.1223	0.0010	0.0034
		(0.0341)	(0.0898)	(0.0068)	(0.0111)	(0.1484)	(0.4530)	(0.0064)	(0.0136)
	$\frac{\tau}{6}$	-0.0010	-0.0034	0.0003	0.0016	-0.0487	0.0276	-0.0036	-0.0050
		(0.3486)	(0.5592)	(0.0858)	(0.1348)	(1.6348)	(2.7145)	(0.3976)	(0.6345)
	$\frac{\tau}{3}$	0.0058	0.0034	-0.0004	0.0017	-0.0241	-0.0498	-0.0018	0.0016
		(0.4146)	(0.6625)	(0.1011)	(0.1631)	(2.0500)	(3.2824)	(0.5155)	(0.8173)
	$\frac{\tau}{2}$	-0.0074	-0.0022	-0.0001	-0.0009	-0.0301	0.0012	-0.0019	-0.0019
		(0.4441)	(0.6955)	(0.1078)	(0.1701)	(2.1809)	(3.4651)	(0.5420)	(0.8549)
	$\frac{2\tau}{3}$	0.0074	0.0069	-0.0007	0.0001	-0.0069	-0.0697	0.0063	-0.0010
		(0.4050)	(0.6476)	(0.1040)	(0.1613)	(2.0679)	(3.2897)	(0.5037)	(0.8103)
	$\frac{5\tau}{6}$	0.0061	0.0203	-0.0012	0.0007	-0.0205	-0.0477	0.0019	0.0109
		(0.3498)	(0.5504)	(0.0862)	(0.1393)	(1.6576)	(2.7661)	(0.3998)	(0.6338)
	$\tau - 1$	-0.0066	-0.0192	-0.0002	-0.0009	-0.0496	-0.2344	-0.0010	-0.0031
		(0.0354)	(0.0816)	(0.0069)	(0.0113)	(0.1683)	(0.8114)	(0.0071)	(0.0141)

Table 1: Biases (Standard Errors) of $\hat{a}^+$

T	a^+	$\tau = 18$				$\tau = 90$			
		$\rho = 1$		$\rho = 4$		$\rho = 1$		$\rho = 4$	
		$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$
180	1	-0.1216	-0.3542	-0.0033	-0.0105	-0.7385	-1.9807	-0.0195	-0.0624
		(0.3834)	(1.0334)	(0.0389)	(0.0679)	(2.1769)	(5.6903)	(0.1058)	(0.2674)
	$\frac{\tau}{6}$	-0.0061	-0.1891	0.0049	0.0088	0.0272	-0.3193	0.0098	-0.0285
		(0.7900)	(1.4824)	(0.1768)	(0.2821)	(4.3256)	(7.7935)	(0.8440)	(1.3914)
	$\frac{\tau}{3}$	0.0189	-0.0348	-0.0024	-0.0041	0.1718	0.3437	-0.0033	-0.0348
		(0.9868)	(1.7575)	(0.2109)	(0.3341)	(5.0251)	(9.3200)	(1.0577)	(1.6962)
	$\frac{\tau}{2}$	0.0106	-0.0224	0.0004	-0.0033	-0.0912	0.0750	0.0075	0.0111
		(1.0240)	(1.9118)	(0.2226)	(0.3573)	(5.2586)	(10.2004)	(1.1088)	(1.7416)
	$\frac{2\tau}{3}$	-0.0117	0.0282	0.0015	0.0047	-0.2101	-0.1555	-0.0021	-0.0322
		(0.9743)	(1.7691)	(0.2098)	(0.3424)	(4.9596)	(9.6431)	(1.0617)	(1.7200)
	$\frac{5\tau}{6}$	0.0155	0.1935	-0.0014	-0.0032	-0.1870	0.6235	0.0103	-0.0444
		(0.7730)	(1.4735)	(0.1825)	(0.2850)	(4.2045)	(7.8993)	(0.8540)	(1.3660)
	$\tau - 1$	0.1112	0.3423	0.0041	0.0086	1.3665	3.0569	-0.0066	0.0308
		(0.3568)	(1.0428)	(0.0382)	(0.0677)	(2.8552)	(6.6974)	(0.1974)	(0.4815)
900	1	-0.0124	-0.0533	-0.0005	-0.0015	-0.0787	-0.2737	-0.0029	-0.0073
		(0.0809)	(0.1890)	(0.0166)	(0.0260)	(0.3204)	(0.9604)	(0.0354)	(0.0604)
	$\frac{\tau}{6}$	0.0096	-0.0045	-0.0011	0.0008	0.0065	0.0711	-0.0009	0.0034
		(0.3204)	(0.5314)	(0.0784)	(0.1242)	(1.5634)	(2.7500)	(0.3739)	(0.5926)
	$\frac{\tau}{3}$	0.0041	0.0028	0.0027	-0.0008	0.0045	-0.0100	0.0034	0.0189
		(0.3835)	(0.6368)	(0.0957)	(0.1471)	(1.9168)	(3.1786)	(0.4665)	(0.7312)
	$\frac{\tau}{2}$	-0.0017	-0.0111	-0.0010	0.0048	0.0014	-0.0224	-0.0059	0.0058
		(0.4087)	(0.6623)	(0.0994)	(0.1601)	(2.0265)	(3.3976)	(0.4892)	(0.7875)
	$\frac{2\tau}{3}$	-0.0023	0.0175	0.0011	-0.0001	-0.0220	-0.0120	-0.0043	-0.0031
		(0.3821)	(0.6366)	(0.0945)	(0.1498)	(1.9350)	(3.2125)	(0.4653)	(0.7346)
	$\frac{5\tau}{6}$	-0.0138	-0.0047	0.0007	-0.0017	-0.0157	-0.0984	-0.0117	-0.0077
		(0.3244)	(0.5315)	(0.0790)	(0.1260)	(1.5820)	(2.7917)	(0.3688)	(0.5907)
	$\tau - 1$	0.0137	0.0452	0.0005	0.0017	0.0623	0.5320	0.0025	-0.0003
		(0.0813)	(0.1784)	(0.0163)	(0.0269)	(0.6503)	(1.6029)	(0.0356)	(0.1033)
2700	1	-0.0026	-0.0103	-0.0001	-0.0005	-0.0187	-0.0594	-0.0001	-0.0017
		(0.0400)	(0.0725)	(0.0096)	(0.0153)	(0.1107)	(0.2658)	(0.0197)	(0.0314)
	$\frac{\tau}{6}$	0.0021	0.0004	-0.0002	-0.0007	0.0273	-0.0222	0.0015	0.0034
		(0.1845)	(0.2947)	(0.0455)	(0.0712)	(0.8646)	(1.4442)	(0.2113)	(0.3366)
	$\frac{\tau}{3}$	-0.0019	-0.0024	0.0001	-0.0017	0.0212	0.0242	0.0013	-0.0045
		(0.2195)	(0.3526)	(0.0540)	(0.0866)	(1.0903)	(1.7440)	(0.2709)	(0.4352)
	$\frac{\tau}{2}$	0.0031	0.0026	-0.0001	0.0005	0.0146	0.0027	0.0001	-0.0030
		(0.2366)	(0.3694)	(0.0573)	(0.0899)	(1.1508)	(1.8328)	(0.2859)	(0.4557)
	$\frac{2\tau}{3}$	-0.0032	-0.0025	0.0001	-0.0000	0.0025	0.0441	-0.0028	-0.0010
		(0.2159)	(0.3453)	(0.0552)	(0.0852)	(1.0980)	(1.7374)	(0.2683)	(0.4312)
	$\frac{5\tau}{6}$	-0.0022	-0.0109	0.0004	-0.0007	0.0177	0.0218	-0.0011	-0.0056
		(0.1847)	(0.2899)	(0.0458)	(0.0736)	(0.8782)	(1.4581)	(0.2117)	(0.3371)
	$\tau - 1$	0.0037	0.0087	0.0002	0.0004	-0.0038	0.0470	0.0005	0.0019
		(0.0394)	(0.0696)	(0.0095)	(0.0152)	(0.2040)	(0.5483)	(0.0199)	(0.0341)

Table 2: Biases (Standard Errors) of $\hat{\varphi}$

| | $\tau = 18$ | | | | $\tau = 90$ | | | |
| | $\rho = 1$ | | $\rho = 4$ | | $\rho = 1$ | | $\rho = 4$ | |
T	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 2.5$
180	0.0096	0.0229	0.0016	0.0068	0.0110	0.0300	0.0037	0.0068
	(0.1060)	(0.1660)	(0.1053)	(0.1672)	(0.1052)	(0.1646)	(0.1052)	(0.1665)
900	0.0024	0.0053	0.0006	0.0011	0.0021	0.0054	0.0010	0.0015
	(0.0470)	(0.0744)	(0.0472)	(0.0746)	(0.0472)	(0.0744)	(0.0472)	(0.0745)
2700	0.0004	0.0016	0.0002	0.0005	0.0005	0.0022	0.0003	0.0001
	(0.0273)	(0.0431)	(0.0272)	(0.0430)	(0.0271)	(0.0431)	(0.0273)	(0.0430)

Table 3: Biases (Standard Errors) of $\hat{\rho}$

value of around $\pm 4\%$ of the period length for $\tau = 18$ and up to (5%,-7%) for $\tau = 90$. These maxima and minima occur at $a^+ = 1$ and $a^+ = \tau - 1$ respectively implying a trend towards central (symmetrical) values that is expectable with low SNR and T. Also the increment in bias for the larger period obeys to the smaller number of waves within the same sample length and disappears as T or SNR increase. For the rest of cases of sample lengths and SNR the bias gets to almost zero (less than 0.5% of the period length).

Two aspects can be appreciated in the performance of $bias_{\hat{\varphi}}$ (Table 2) as compared with $bias_{\hat{a}^+}$. Firstly, when a trend becomes apparent it does in opposite direction and so it grows with the value of a^+, which implies a negative covariance between these two estimators. Secondly, the values are around one half the biases of $\hat{a}^+$ in all situations, indicating a better performance in the estimation of this parameter.

The standard errors in Tables 1 and 2 decrease along with σ_ε^2 and τ and as the amplitude and the sample length increase. Specifically when comparing the results across different SNR and T, the factor $\frac{\tau \sigma_\varepsilon}{\rho \sqrt{T}}$ constitutes a steady component of the standard errors of both $\hat{a}^+$ and $\hat{\varphi}$ so from the Monte Carlo analysis herein it can be drawn that these estimators seem to behave consistently with a rate of convergence of $T^{-\frac{1}{2}}$. The other main aspect of the SEs is a clearly negative dependence on the degree of asymmetry. Finally, focusing on the magnitudes, in the case of a^+ and in the lowest SNR and small sample length $T = 180$ the standard error reaches a maximum value of 20% of the period length (21% for $\tau = 90$) when $a^+ = \frac{\tau}{2}$ that decreases down to 10%-11% (12%-14%) for $a^+ = 1$ and $a^+ = \tau - 1$. In the most favourable situation of $\rho = 4$, $\sigma_\varepsilon^2 = 1$ and $T = 2700$ these values have decreased to 0.59% (0.60%) for $a^+ = \frac{\tau}{2}$ and 0.02%-0.03 (0.005%-0.007%) for $a^+ = 1$ and $a^+ = \tau - 1$. Regarding $\hat{\varphi}$, the behaviour of the standard error is completely parallel to what has just been described with the added fact that the values of $SE_{\hat{\varphi}}$ are one half the corresponding of $SE_{\hat{a}^+}$, as happened to the bias.

Wave	Peaks	Troughs	Months(τ)
1	Nov. 1948	Oct. 1949	57
2	Jul. 1953	May. 1954	49
3	Aug. 1957	Apr. 1958	32
4	Apr. 1960	Feb. 1961	116
5	Dec. 1969	Nov. 1970	47
6	Nov. 1973	Mar. 1975	74
7	Jan. 1980	Jul. 1980	18
8	Jul. 1981	Nov. 1982	108
9	Jul. 1990	Mar. 1991	128
10	Mar. 2001	Nov. 2001	80

Table 4: NBER Business Cycle Peaks and Troughs

Table 3 displays the results of estimation of the amplitude. A recurrent positive bias (that goes to 0 as T increases) has been obtained which implies an overestimation of this parameter for finite samples. This is attributable to the chosen initial estimate $\hat{\rho}_0$ that relates to the explained variance of an unrestricted τ-perioded model and therefore is expectably larger than the true value of the amplitude of the asymmetric cycle. Finally, it can be clearly appreciated that $SE_{\hat{\rho}} \approx \sigma_{\varepsilon}\sqrt{\frac{2}{T}}$, showing that the estimator of the amplitude of asymmetric cycles behaves asymptotically as in the symmetric case (Zhou and Giannakis, 1995).

7 Empirical applications

7.1 The business cycle

In this section we will deal with a monthly version of the US seasonally adjusted unemployment rate (UR) compiled by the Federal Reserve Bank of St. Louis. With the purpose of isolating the asymmetric cyclical behaviour the sample was limited to complete periods of booms and recessions so following the NBER Busyness Cycle reference dates (Table 4) the sample spans from November 1948 to November 2007 with a length of $T = 709$ months. The series, displayed in Figure 4(a), is characterized by a cycle that evolves over a long-term pattern. The long-term component (Figure 4(b)) and the short-term (cyclical) component (Figure 4(c)) were separated using a Hodrick-Prescott filter with smoothing parameter $\vartheta = 120000$ (see Maravall and del Rio, 2001).
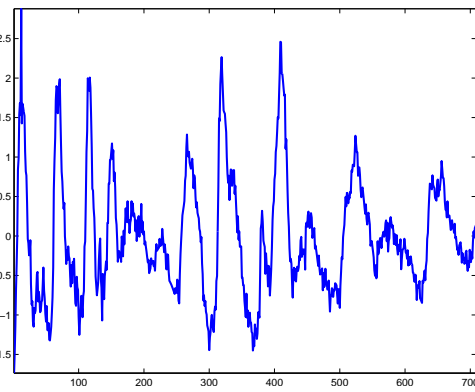
Figure 5 depicts the correlogram and the periodogram of the extracted
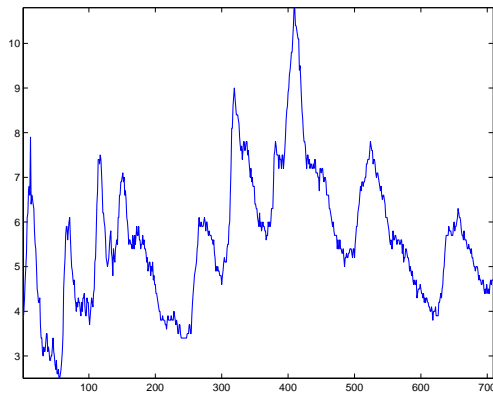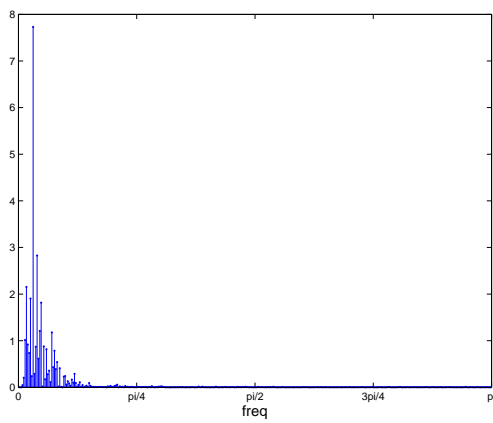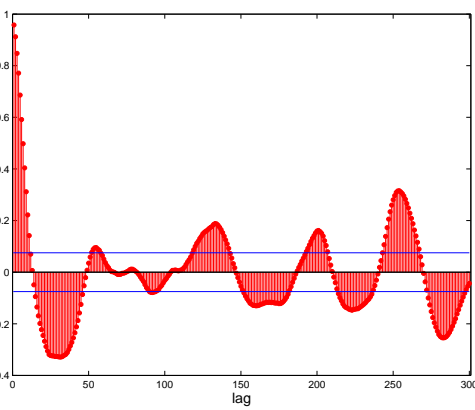
(a) UR
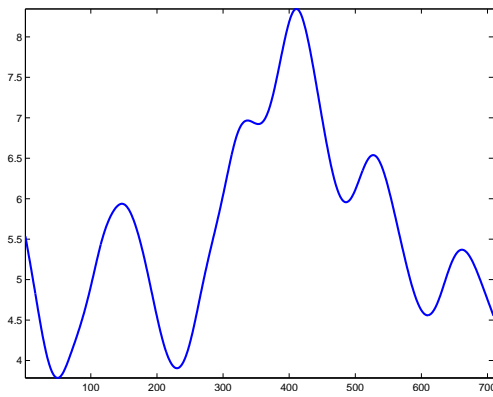
(b) Long-Term Component

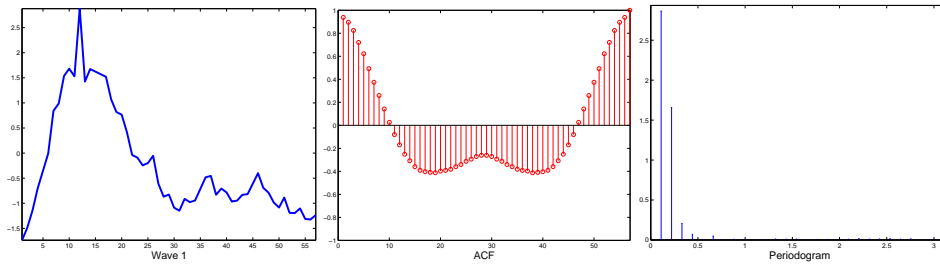(c) Cyclical Component

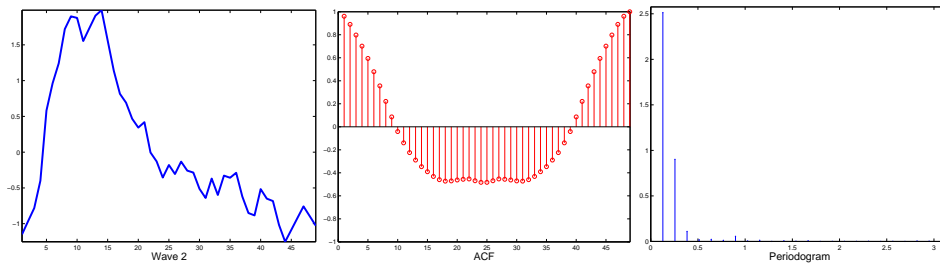Figure 4: US SA Unemployment Rate

(a) Periodogram
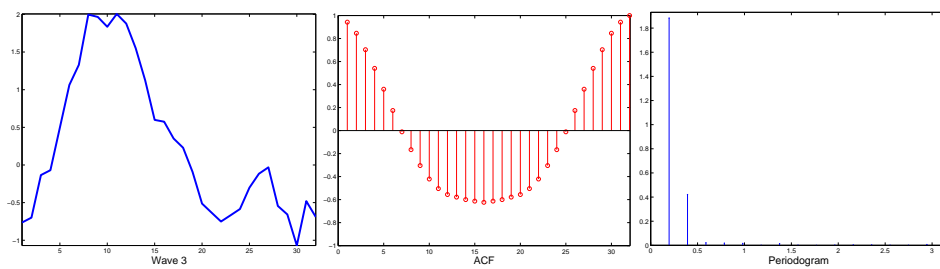

(b) Correlogram

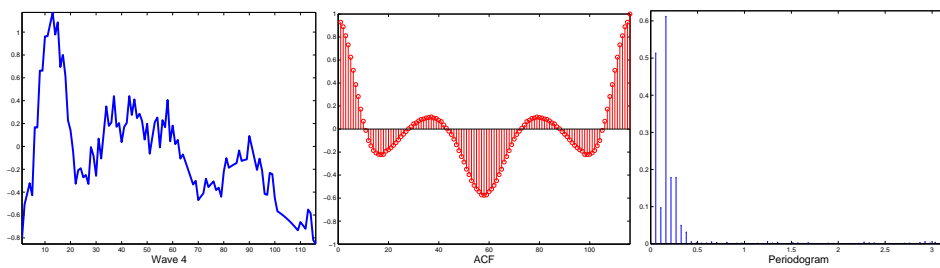Figure 5: Periodogram and Correlogram of UR short-term component
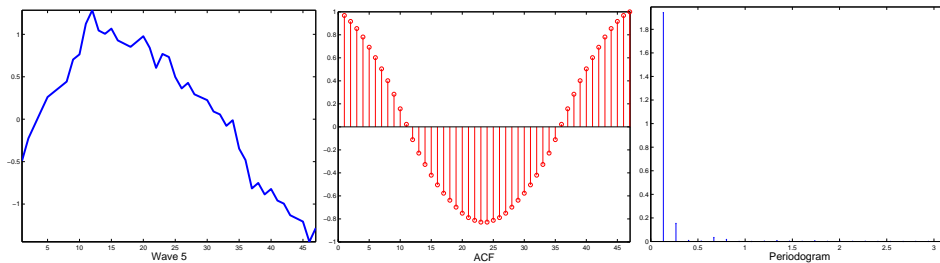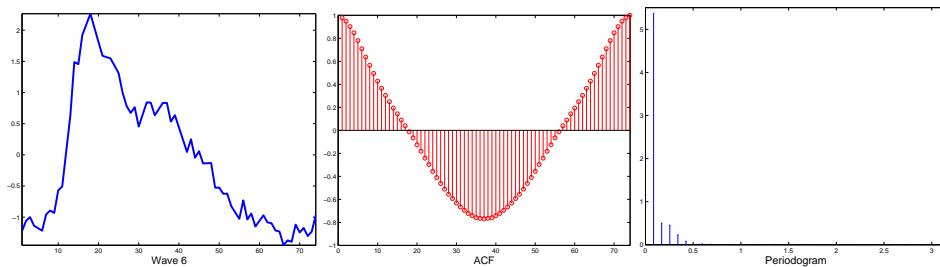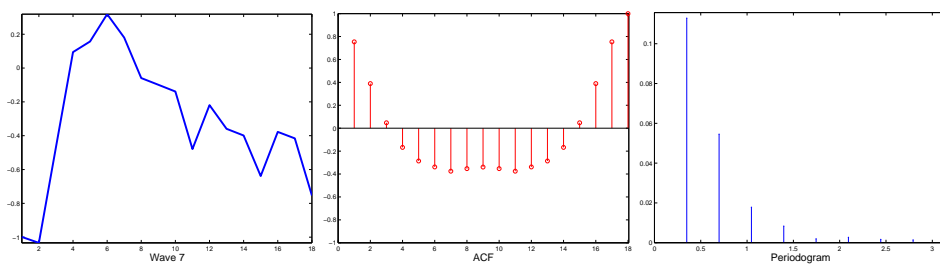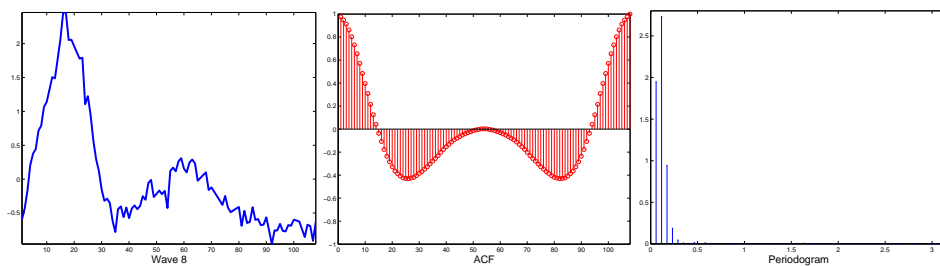
(a) Wave 1

(b) Wave 2

(c) Wave 3

(d) Wave 4

Figure 6: UR Waves

(e) Wave 5

(f) Wave 6

(g) Wave 7

(h) Wave 8

Figure 6: UR Waves (Cont.)

(i) Wave 9

(j) Wave 10
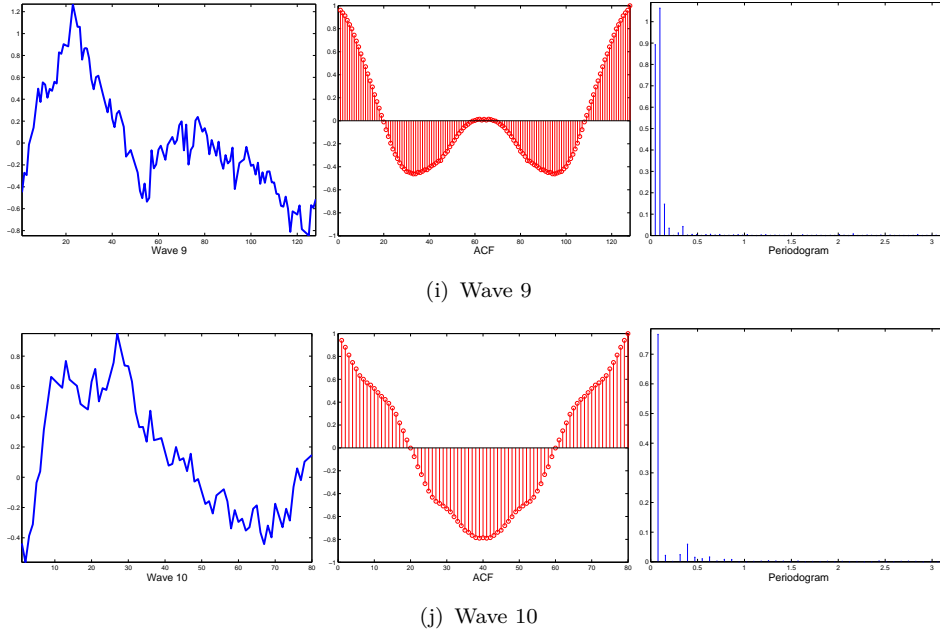
Figure 6: UR Waves (Cont.)

short-term component. No signs of cyclical asymmetry can be deduced from these graphs due to the variability in the period of the consecutive fluctuations. The series was then divided in trough-to-trough waves taking into account the dates of Table 4. The analysis of the correlogram and periodogram of each wave (Figure 6) allows to search for individual cyclical asymmetric patterns. Regarding this, the correlograms and periodograms of waves 1, 2, 3, 5, 6, 7 and 10 (Figures 6(a) to 6(c), 6(e) to 6(g) and 6(j)) show signs of cyclical asymmetry. In these cases the ACF troughs are significantly shorter than the peaks and there appears the corresponding decreasing shape in the multiples of λ_i^{asy}, $i = 1, \ldots, 10$, on the periodograms. By contrast, the correlograms and periodograms of waves 4, 8 and 9 (Figures 6(d), 6(h) and 6(i)) suggest the presence of patterns different to cyclical asymmetry, specifically the periodogram ordinates at $j\lambda_i^{asy}$, $j > 1$ indicate that in these series there are predominant sub-cycles of period smaller than or equal to $\tau_i/2$ that account for a larger portion of variance than the τ_i-perioded oscillation.

Both symmetric and asymmetric models were fitted to each wave of the cyclical component. In this process the phase parameter φ was restricted as $\varphi_i = a_i^-$ ($\varphi_i = \tau_i/2$ in the symmetric models) so the estimation fitted trough-to-trough waves. The results are displayed in Table 5[3].

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[3]The waves 4, 8 and 9 have been included in Table 5 for completeness although taking into account the conclusions drawn from their ACFs and periodograms in Figure 6 their

Wave	$\hat{a}^+$	$\hat{a}^-$	$\hat{a}^+/\tau$	$\hat{\rho}^{\mathbf{asy}}$	$\hat{\rho}^{\mathbf{sym}}$	$\hat{\sigma}_{a^+}$	$S\hat{N}R$
1	8.8842**[1]	48.1158	0.1559	1.2542	0.13154	1.06	2.3937
2	7.8106**	41.1894	0.1594	1.259	0.27878	0.685	5.0158
3	7.8726**	24.1274	0.246	1.2424	0.6363	0.712	4.1938
4	9.7**	106.3	0.0836	0.4648	0.10478	1.567	1.2989
5	11.4543**	35.5457	0.2437	1.0291	0.72644	0.627	7.8957
6	18.9566**	55.0434	0.2562	1.4338	0.89768	0.761	8.7143
7	6.0711*	11.9289	0.3373	0.441	0.36344	0.886	1.8396
8	6.5918**	101.4082	0.061	0.8622	-0.16104	1.4122	1.1165
9	9.2228**	118.7772	0.0721	0.5509	0.020513	1.2415	2.0525
10	11.7281**	68.2719	0.1466	0.492	0.14585	0.596	3.9356

[1] * and ** indicate rejection of $H_0 \equiv a^+ = \frac{\tau}{2}$ against $H_a \equiv a^+ < \frac{\tau}{2}$ at the 1% and 0.1% significance level respectively.

Table 5: UR Estimation Results

The results of Table 5 show that in no wave of the business cycle the recession (in economic sense) lasted for longer than one third the length of the fluctuation. The most common duration for this stage of the cycle lies in the range of around 15% to 25% of the wave period (waves 1, 2, 3, 5, 6 and 10) and only the shortest fluctuation (wave 7, $\tau_7 = 18$) surpasses this interval. There is also a clear difference in the two estimated amplitudes for each wave and for example the symmetric restriction practically impedes in waves 1 or 2 the estimation of this parameter.

The null hypothesis of symmetry $H_0 : a_i^+ = \frac{\tau_i}{2}$ against the alternative $H_a : a_i^+ < \frac{\tau_i}{2}$ was tested in all these waves using the t-ratio

$$\hat{t}_i = \frac{\hat{a}_i^+ - \frac{\tau_i}{2}}{\hat{\sigma}_{\hat{a}_i^+}}, \tag{17}$$

where $\hat{\sigma}_{\hat{a}_i^+}$ is calculated as in (16) for only the two parameters a^+ and ρ and where the parametric values were substituted by their corresponding estimations $\hat{\sigma}_{\varepsilon_i}^2 = \frac{RSS(\hat{a}_i^+, \hat{\rho}_i)}{\tau_i - 2}$, $\hat{a}_i^+$ and $\hat{\rho}_i$. Taking into account the conclusions drawn from the Monte Carlo analysis we can, with sufficient confidence, approximate the distribution of this t-ratio by a Gaussian standard distribution. Following this approach, Table 5 shows that the hypothesis of symmetry is rejected in all waves at the 0.1% or 1% significance level.

Regarding the long waves 4, 8 and 9, their periodograms in Figure 5 report the presence of 3, 2 and 2 predominant secondary fluctuations respectively and for the analysis of these, the turning points between them must

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results won't be considered here. The analysis of the asymmetries of the subcycles of these waves is presented in Table 6.

Wave	$\hat{\mathbf{a}}^+$	$\hat{\mathbf{a}}^-$	$\hat{\mathbf{a}}^+/\tau$	$\hat{\rho}^{\mathbf{asy}}$	$\hat{\rho}^{\mathbf{sym}}$	$\hat{\sigma}_{a^+}$	$S\hat{N}R$
4.1	12.9056	14.0944	0.4780	0.8008	0.7983	0.4536	11.7389
4.2	11.1939**	35.8061	0.2382	0.2752	0.2219	1.3490	1.6805
4.3	8.3398**	33.6602	0.1986	0.3115	0.1716	0.9380	2.7259
8.1	15.9460**	22.0540	0.4196	1.3620	1.3092	0.4291	18.0179
8.2	12.8285**	57.1715	0.1833	0.4049	0.1883	1.1551	2.8158
9.1	21.5121**	32.4879	0.3984	0.6021	0.5676	0.7746	7.7322
9.2	10.4482**	63.5518	0.1412	0.3208	0.1641	1.0187	3.1017

[1] ** indicate rejection of $H_0 \equiv a^+ = \frac{\tau}{2}$ against $H_a \equiv a^+ < \frac{\tau}{2}$ at the 0.1% significance level respectively.

Table 6: Estimation Results of UR secondary fluctuations in the waves 4, 8 and 9

be determined. In order to do so, a number of moving average filters with different weighting schemes and different window lengths (upto 11) were used and the turning points were selected as the points where most filtered series showed a change in slope. Specifically, in wave 4 the first secondary fluctuation was found to last until July 1962 with a length of 27 observations, the second one from this point to June 1966 with 47 observations and then the last one has a length of 42 observations. In wave 8, the turning point takes place in September 1984 and the two subfluctuations last 38 and 70 observations respectively. Finally, the turning point between the two subfluctuations of wave 9 is located in January 1995 and the length of the wave is distributed as 54 months for the first subfluctuation and 74 for the second one. The results of the estimation of the half-periods and amplitudes of these subcycles (Table 6) show that in most cases (except in the subcycle 4.1) the contractionary stage is shorter than the expansionary stage. It can also be seen in the three waves 4, 8 and 9 that the first subfluctuation is the most balanced (over 39% of their total lengths).

Figure 7 shows the ten waves together with both asymmetric and symmetric cyclical models and Figure 8 the corresponding fits for the secondary fluctuations of waves 4, 8 and 9. The asymmetric fit seems a reliable and better model for all waves 1, 2, 3, 5, 6, 7 and 10 and all subwaves 4.2, 4.3, 8.1, 8.2, 9.1 and 9.2.

(a) Wave 1 (b) Wave 2 (c) Wave 3

(d) Wave 4 (e) Wave 5 (f) Wave 6

(g) Wave 7 (h) Wave 8 (i) Wave 9

(j) Wave 10
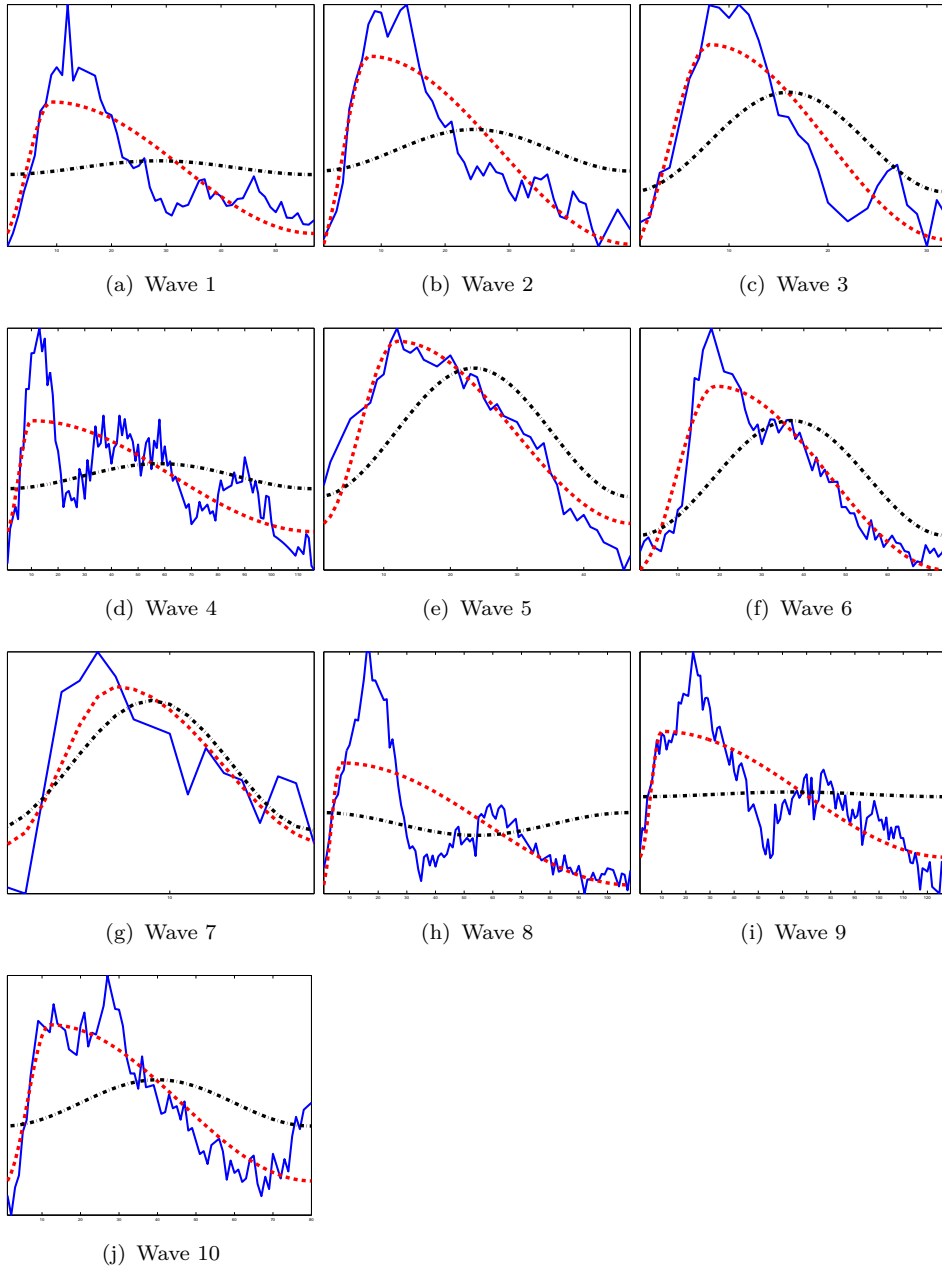
Figure 7: UR Waves with symmetric (dotted) and asymmetric (dashed) fitted models

(a) Subwave 4.1   (b) Subwave 4.2   (c) Subwave 4.3
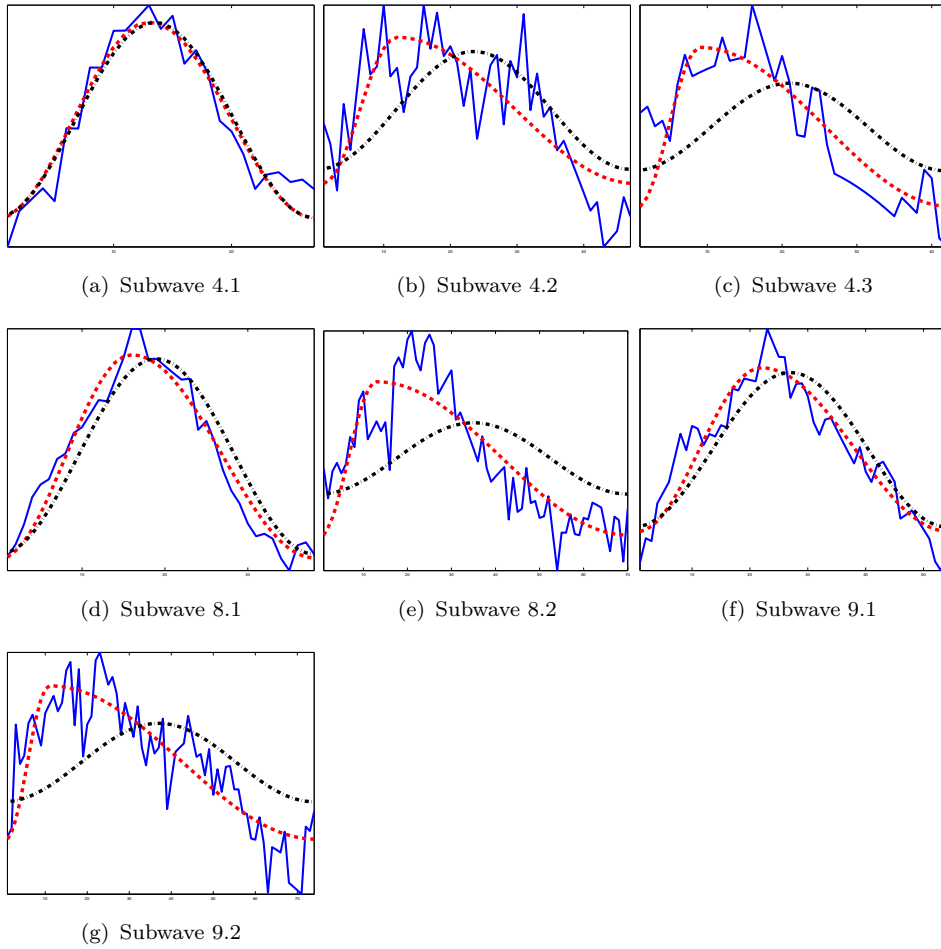
(d) Subwave 8.1   (e) Subwave 8.2   (f) Subwave 9.1

(g) Subwave 9.2

Figure 8: UR secondary fluctuations of waves 4, 8 and 9 with symmetric (dotted) and asymmetric (dashed) fitted models

(a) Series   (b) Correlogram
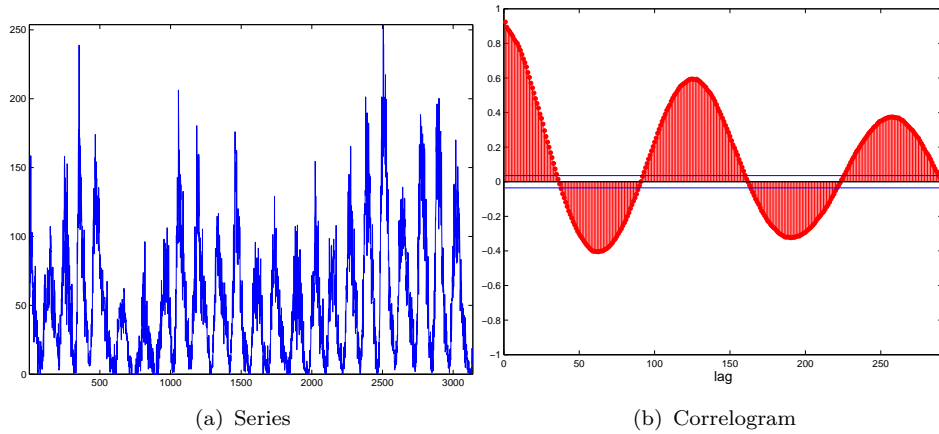
Figure 9: Sunspot Index

7.2 The sunspot index

As is common usage in the literature of cyclical time series analysis, a similar study of the sunspot index was carried out. The sunspots series has been considered as frequency asymmetric by many authors such as Nesme-Ribes et al. (1994), Hathaway et al. (1994), Polygiannakis et al. (1996) and Passos and Lopes (2008). In our case we will work with a version of the series that comprises monthly observations of the sunspots from March 1755 to December 2008, spanning through all 23 complete solar cycles thoroughly recorded so far with a total of $T = 3046$ observations. The series is displayed in Figure 9(a). The correlogram of the series (Figure 9(b)) shows clear evidence of cyclical asymmetry.

Table 7 displays the results of estimation of the degree of asymmetry in each individual solar cycle. According to these, only two cycles possess longer expansions than recessions (cycles 1 and 7), one cycle (number 6) is substantially symmetric and the rest of cycle expansions move within a range of 20% to 40% of the cycle duration. The null hypothesis of symmetry was tested as in the previous application, two-sidedly in this case as both types of asymmetry may occur. The results of these tests indicate that the hypothesis of symmetry is rejected in all the cycles except number 6 at the 0.1% significance level.

Figure 10 depicts the solar cycles together with both asymmetric and symmetric models.

Solar Cycle	$\hat{a}^+$	$\hat{a}^-$	$\hat{a}^+/\tau$	$\hat{\rho}^{\textbf{asy}}$	$\hat{\rho}^{\textbf{sym}}$	$\hat{\sigma}_{a^+}$	$\hat{SNR}$
1	76.514**[1]	59.486	0.563	30.458	29.726	2.108	2.699
2	42.189**	65.811	0.391	45.433	42.266	2.036	2.223
3	37.186**	73.814	0.335	67.979	55.353	1.279	5.419
4	33.165**	130.835	0.202	60.209	28.059	1.064	8.367
5	58.435**	92.565	0.387	24.398	22.644	1.529	5.489
6	72.571	76.429	0.487	20.421	20.396	2.719	1.804
7	80.888**	45.112	0.642	33.561	30.087	1.682	3.668
8	38.892**	77.108	0.335	61.069	50.445	1.438	4.477
9	58.762**	90.238	0.394	48.258	44.469	2.074	2.965
10	50.946**	84.054	0.377	40.750	36.932	1.558	4.681
11	38.487**	102.513	0.273	60.374	38.427	1.419	4.975
12	41.784**	93.216	0.310	33.047	27.158	1.786	3.240
13	30.841**	112.159	0.216	39.597	20.796	1.249	5.551
14	45.563**	92.437	0.330	31.464	26.933	2.087	2.511
15	47.980**	72.020	0.400	42.386	39.459	1.804	3.169
16	44.167**	76.833	0.365	35.860	32.544	1.590	3.976
17	51.111**	73.889	0.409	51.979	49.457	1.515	4.712
18	46.904**	75.096	0.384	71.756	66.292	1.254	6.578
19	39.971**	86.029	0.317	95.690	75.431	0.980	10.174
20	46.142**	93.858	0.330	48.976	41.189	1.287	6.685
21	43.874**	79.126	0.357	77.309	68.363	1.061	8.985
22	36.180**	79.820	0.312	76.900	61.515	1.049	8.098
23	45.819**	105.181	0.303	57.319	43.505	1.297	6.790

[1] ** indicates rejection of $H_0 \equiv a^+ = \frac{\tau}{2}$ against $H_a \equiv a^+ \neq \frac{\tau}{2}$ at the 0.1% significance level

Table 7: Sunspot Index Estimation Results
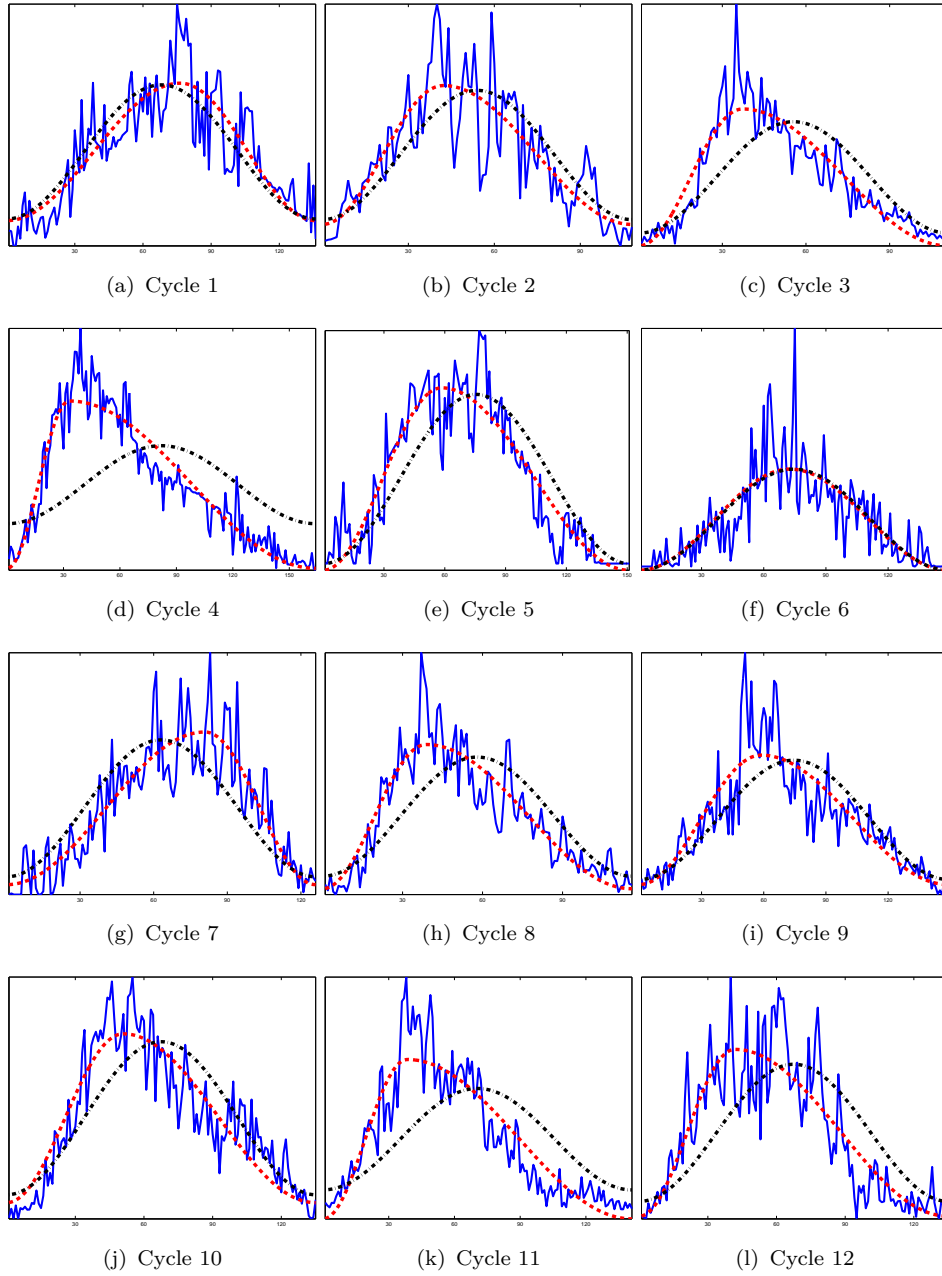
(a) Cycle 1   (b) Cycle 2   (c) Cycle 3

(d) Cycle 4   (e) Cycle 5   (f) Cycle 6

(g) Cycle 7   (h) Cycle 8   (i) Cycle 9

(j) Cycle 10   (k) Cycle 11   (l) Cycle 12

Figure 10: Sunspot Cycles with symmetric (dotted) and asymmetric (dashed) fitted models

(m) Cycle 13   (n) Cycle 14   (o) Cycle 15

(p) Cycle 16   (q) Cycle 17   (r) Cycle 18

(s) Cycle 19   (t) Cycle 20   (u) Cycle 21

(v) Cycle 22   (w) Cycle 23
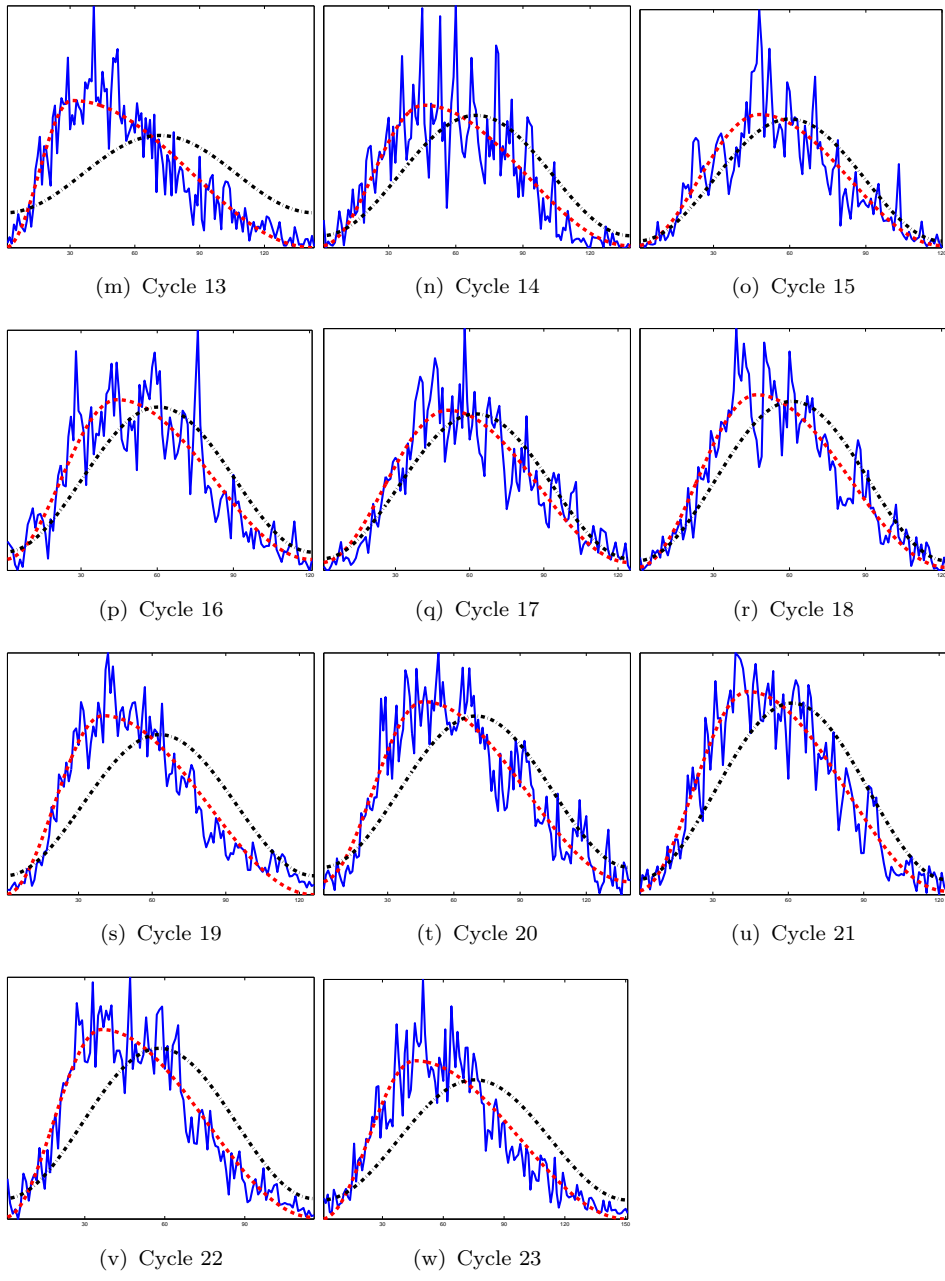
Figure 10: Sunspot Cycles with symmetric (dotted) and asymmetric (dashed) fitted models (Cont.)

8   Conclusions

In this work I have proposed a simple deterministic cycle with an asymmetric behaviour in frequency, which can adequately describe different velocities of adjustment to increases and decreases of the series.

This is a common phenomenon for example in macroeconomic series where the periods of expansion have a different length than the periods of recession. An analytical expression for both the periodogram and the ACV of such cycles has been derived, evidencing a distinct behaviour which can be used to discriminate between symmetric and asymmetric cycles.

In particular the main feature of the correlogram of frequency asymmetric cycles consists in a cosinoidal wave that evolves with less deep troughs than peaks are high, a difference in amplitude that becomes more evident as one of the two asymmetric frequencies approaches π.

In the periodogram, the frequency asymmetry appears as a monotonically decreasing shape at least in the four first Fourier frequencies. Again, this feature becomes more prominent as $\max(\lambda^+, \lambda^-) \to \pi$.

This last case implies that the cycle covers all the period of expansion or contraction in one single step and constitutes the pole of *complete frequency asymmetry* in the behaviour of second-order moments whereas the other pole is represented by the symmetric cycle.

A Non-Linear Least Squares procedure of estimation of the asymmetric frequencies, the amplitude and the phase of the cycle has also been proposed and its finite sample performance in an extensive grid of parametric values of the model checked. The results suggest that the procedure performs well for a wide range of sample lengths and SNRs.

The empirical application focuses on the US Unemployment Rate and on the Sunspot Index. Sustainable evidence of asymmetry in most fluctuations of the business cycle as expressed by the US Unemployment Rate is reported by the proposed statistical procedures. Specifically, using the NBER reference dated waves from November 1948, the ACFs and periodograms of waves 1, 2, 3, 5, 6, 7 and 10 show signs of cyclical asymmetry. These graphs also show that in the longest fluctuations (waves 4, 8 and 9) the serial dependence of the series obeys to the existence of predominant sub-cycles. The estimation of the asymmetric frequencies of the Unemployment Rate yields duration percentages for the recessions significantly different from 50 per cent of the wave length, between 15% and 25% in the fluctuations 1, 2, 3, 5, 6 and 10 of the business cycle and 33% in the seventh one. In the case of the longest business cycle waves 4, 8 and 9 the analysis shows that they are composed by an initial almost symmetric large fluctuation followed by one or two highly asymmetric smaller ones.

In the case of the Sunspot Index, it has been shown that the correlogram of the series holds a clear pattern of cyclical asymmetry. The estimation of the asymmetric frequencies confirmed that in most fluctuations of the series (except in solar cycles 1, 6 and 7) the expansions are shorter than the contractions (between 20 to 40 per cent of the cycle length).

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A   Appendix

The proofs of Proposition 1 and 2 use $\varphi = 0$. The results can be immediately generalized for any positive integer φ as x_t for $1 \leq t \leq \tau$ will evaluate at identical points at different positions.

A.1   Proof of Proposition 1

Let x_t be as defined in (1) and $\varphi = 0$. Then

$$
M_x = \frac{\rho}{\tau} \left\{ \underbrace{\sum_{t=1}^{a^-} \cos\left(\lambda^- t\right)}_{M_{\lambda^-}} + \underbrace{\sum_{t=a^-+1}^{\tau} -\cos\left[\lambda^+\left(t - a^-\right)\right]}_{M_{\lambda^+}} \right\}.
$$

Now by Equation 1342.2 in Gradstein and Ryzhik (2000) (GR)

$$M_{\lambda^-} = \frac{\cos\left(\dfrac{\pi + \lambda^-}{2}\right)\sin\dfrac{\pi}{2}}{\sin\dfrac{\lambda^-}{2}} = -1$$

and

$$M_{\lambda^+} = -\sum_{k=1}^{a^+}\cos\left(\lambda^+ k\right) = 1.$$

Then

$$M_x = \frac{\rho}{\tau}(-1 + 1) = 0$$

Now

$$S_x^2 = \frac{\rho^2}{\tau}\left\{\underbrace{\sum_{t=1}^{a^-}\cos^2\left(\lambda^- t\right)}_{S_{\lambda^-}} + \underbrace{\sum_{t=a^-+1}^{\tau}\cos^2\left[\lambda^+\left(t - a^-\right)\right]}_{S_{\lambda^+}}\right\}$$

By Eq. 1351.2 in GR

$$S_{\lambda^-} = \frac{\pi}{2\lambda^-} + \frac{\cos\left(\pi + \lambda^-\right)\sin\pi}{2\sin\lambda^-} = \frac{\pi}{2\lambda^-}$$

and

$$S_{\lambda^+} = \sum_{k=1}^{a^+}\cos^2\left(\lambda^+ k\right) = \frac{\pi}{2\lambda^+}.$$

Then

$$S_x^2 = \frac{\rho^2}{\tau}\left(\frac{\pi}{2\lambda^-} + \frac{\pi}{2\lambda^+}\right) = \frac{\rho^2}{\tau}\frac{\pi}{\lambda^{asy}} = \frac{\rho^2}{2}.$$

However, if $\lambda_{max} = \pi \Rightarrow S_{\lambda_{max}} = \cos^2\left(\pi\right) = 1$ and then

$$S_x^2 = \frac{\rho^2}{\tau}\left(\frac{\pi}{2\lambda_{min}} + 1\right) = \frac{\rho^2}{2}\left(1 + \frac{1}{\tau}\right).$$

A.2   Proof of Proposition 2

Let x_t be as defined in (1) and $\varphi = 0$. In the products of the autocovariance function of x_t both phases of the cycle are crossed with themselves and with each other at lags determined by k. This produces three different sums of four components such as:

1. $k \leq a_{min}$: Then

$$
s(k) = \frac{\rho^2}{\tau} \left\{ \underbrace{\sum_{j=1}^{a_{max}-k} \cos\left[\lambda_{min}\left(j+k\right)\right] \cos\left(\lambda_{min}j\right)}_{S_{11}(k)} \right.
$$

$$
+ \underbrace{\sum_{j=1}^{k} \cos\left[\lambda_{min}\left(k-j\right)\right] \cos\left(\lambda_{max}j\right)}_{S_{12}(k)}
$$

$$
+ \underbrace{\sum_{j=1}^{a_{min}-k} \cos\left(\lambda_{max}j\right) \cos\left[\lambda_{max}\left(j+k\right)\right]}_{S_{13}(k)}
$$

$$
\left. + \underbrace{\sum_{j=1}^{k} \cos\left(\lambda_{min}j\right) \cos\left[\lambda_{max}\left(k-j\right)\right]}_{S_{14}(k)} \right\} \tag{18}
$$

2. $a_{min} < k \leq a_{max}$:

$$
s(k) = \frac{\rho^2}{\tau} \left\{ \underbrace{\sum_{j=1}^{a_{max}-k} \cos\left(\lambda_{min}j\right) \cos\left[\lambda_{min}\left(j+k\right)\right]}_{S_{21}(k)} \right.
$$

$$
+ \underbrace{\sum_{j=1}^{a_{min}} \cos\left[\lambda_{min}\left(k-j\right)\right] \cos\left(\lambda_{max}j\right)}_{S_{22}(k)}
$$

$$
+ \underbrace{\sum_{j=1}^{k-a_{min}} \cos\left[\lambda_{min}\left(\tau-k+j\right)\right] \cos\left(\lambda_{min}j\right)}_{S_{23}(k)}
$$

$$
\left. + \underbrace{\sum_{j=1}^{a_{min}} \left[-\cos\left(\lambda_{max}j\right)\right] \cos\left[\lambda_{min}\left(k-a_{min}+j\right)\right]}_{S_{24}(k)} \right\} \tag{19}
$$

3. $a_{max} < k \leq \tau$:

$$s(k) = \frac{\rho^2}{\tau} \left\{ \underbrace{\sum_{j=1}^{\tau-k} \cos(\lambda_{min}j)\{-\cos[\lambda_{max}(k-a_{max}+j)]\}}_{S_{31}(k)} \right.$$

$$+ \underbrace{\sum_{j=1}^{k-a_{min}} \cos(\lambda_{min}j)\cos[\lambda_{min}(j+\tau-k)]}_{S_{32}(k)}$$

$$+ \underbrace{\sum_{j=1}^{\tau-k} [-\cos(\lambda_{max}j)]\cos[\lambda_{min}(j-a_{min}+k)]}_{S_{33}(k)}$$

$$\left. + \underbrace{\sum_{j=1}^{k-a_{max}} \cos[\lambda_{max}(j+\tau-k)]\cos(\lambda_{max}j)}_{S_{34}(k)} \right\}$$

Let $k_1 = \frac{\tau}{2} - m$ and $k_2 = \frac{\tau}{2} + m$. Then $S_{21}(k_1) = S_{23}(k_2)$, $S_{22}(k_1) = S_{24}(k_2)$, $S_{11}(k_1) = S_{32}(k_2)$, $S_{12}(k_1) = S_{33}(k_2)$, $S_{13}(k_1) = S_{34}(k_2)$ and $S_{14}(k_1) = S_{31}(k_2)$ and therefore the autocovariance function is symmetric with respect to $\frac{\tau}{2}$. We concentrate now on $k \leq \frac{\tau}{2}$.

From (18)

$$S_{11}(k) = \frac{1}{2}\sum_{j=1}^{a_{max}-k}[\cos(2\lambda_{min}j + \lambda_{min}k) + \cos(\lambda_{min}k)]$$

and by Eq. 1.343.1 in GR we get to

$$S_{11}(k) = \frac{1}{2}\left\{\frac{\cos[\lambda_{min}(a_{max}+1)]\sin[\lambda_{min}(a_{max}-k)]}{\sin\lambda_{min}} + (a_{max}-k)\cos(\lambda_{min}k)\right\}$$

$$= \frac{1}{2}\left[\frac{-\cos\lambda_{min}\sin(\lambda_{min}k)}{\sin\lambda_{min}} + (a_{max}-k)\cos(\lambda_{min}k)\right] = s_1(a_{max}, a_{min}, k).$$
(20)

Also

$$S_{12}(k) = \frac{1}{2}\left\{\sum_{j=1}^{k}\cos[\lambda_{min}k + j(\lambda_{max}-\lambda_{min})] + \sum_{j=1}^{k}\cos[\lambda_{min}k - j(\lambda_{max}-\lambda_{min})]\right\}$$

and again by Eq. 1.343.1 in GR

$$S_{12}(k) = \frac{1}{2} \left\{ \frac{\cos\left[(k-1)\frac{\lambda_{max}+\lambda_{min}}{2} + \lambda_{max}\right] \sin\left[\frac{k(\lambda_{max}-\lambda_{min})}{2}\right]}{\sin\left(\frac{\lambda_{max}-\lambda_{min}}{2}\right)} \right.$$
$$\left. + \frac{\cos\left[(k-1)\frac{\lambda_{max}-\lambda_{min}}{2} + \lambda_{max}\right] \sin\left[\frac{k(\lambda_{max}+\lambda_{min})}{2}\right]}{\sin\left(\frac{\lambda_{max}+\lambda_{min}}{2}\right)} \right\}$$

$$= \frac{1}{2} \left\{ \frac{\sin\left(k\lambda_{max} + \frac{\lambda_{max}-\lambda_{min}}{2}\right) - \sin\left(k\lambda_{min} + \frac{\lambda_{max}-\lambda_{min}}{2}\right)}{\sin\left(\frac{\lambda_{max}-\lambda_{min}}{2}\right)} \right.$$
$$\left. + \frac{\sin\left(k\lambda_{max} + \frac{\lambda_{max}+\lambda_{min}}{2}\right) - \sin\left(-k\lambda_{min} + \frac{\lambda_{max}+\lambda_{min}}{2}\right)}{\sin\left(\frac{\lambda_{max}+\lambda_{min}}{2}\right)} \right\}$$

$$= \frac{1}{2\sin\left(\frac{\lambda_{max}-\lambda_{min}}{2}\right)\sin\left(\frac{\lambda_{max}+\lambda_{min}}{2}\right)} \left[\sin\left(k\lambda_{max} + \frac{\lambda_{max}-\lambda_{min}}{2}\right)\sin\left(\frac{\lambda_{max}+\lambda_{min}}{2}\right) \right.$$
$$- \sin\left(k\lambda_{min} + \frac{\lambda_{max}-\lambda_{min}}{2}\right)\sin\left(\frac{\lambda_{max}+\lambda_{min}}{2}\right)$$
$$+ \sin\left(k\lambda_{max} + \frac{\lambda_{max}+\lambda_{min}}{2}\right)\sin\left(\frac{\lambda_{max}-\lambda_{min}}{2}\right)$$
$$\left. - \sin\left(-k\lambda_{min} + \frac{\lambda_{max}+\lambda_{min}}{2}\right)\sin\left(\frac{\lambda_{max}-\lambda_{min}}{2}\right) \right]$$

and by successive operations we get to

$$S_{12}(k) = \frac{1}{\cos\lambda_{min} - \cos\lambda_{max}} \left\{ 2\cos(k\lambda_{max})\cos\lambda_{min} \right.$$
$$+ 2\cos(k\lambda_{min})\cos\lambda_{max} - 2\cos\left[(k+1)\lambda_{max}\right]$$
$$\left. -2\cos\left[(k-1)\lambda_{min}\right] \right\} = s_2\left(a_{max}, a_{min}, k\right). \tag{21}$$

Now
$$S_{13}(k) = s_1\left(a_{min}, a_{max}, k\right) \tag{22}$$

and $S_{14}(k) = s_2\left(a_{min}, a_{max}, k\right)$ where s_1 and s_2 are as defined in (20) and (21) respectively. Then

$$S_{12}(k) + S_{14}(k) = \frac{\sin(k\lambda_{max})\sin\lambda_{max} - \sin(k\lambda_{min})\sin\lambda_{min}}{\cos\lambda_{min} - \cos\lambda_{max}}$$

and so for $k \leq a_{min}$

$$
\begin{aligned}
s(k) &= \frac{\rho^2}{\tau} \left[S_{11}(k) + S_{12}(k) + S_{13}(k) + S_{14}(k) \right] \\
&= \frac{\rho^2}{\tau} \left[\frac{a_{max} - k}{2} \cos\left(\lambda_{min} k\right) + \frac{a_{min} - k}{2} \cos\left(\lambda_{max} k\right) \right. \\
&\quad - \frac{\cot \lambda_{max} \sin\left(\lambda_{max} k\right)}{2} - \frac{\cot \lambda_{min} \sin\left(\lambda_{min} k\right)}{2} \\
&\quad \left. + \frac{\sin\left(\lambda_{max} k\right) \sin \lambda_{max} - \sin\left(\lambda_{min} k\right) \sin \lambda_{min}}{\cos \lambda_{min} - \cos \lambda_{max}} \right].
\end{aligned}
\tag{23}
$$

Now from (19) $S_{21}(k) = S_{11}(k)$. Also

$$
\begin{aligned}
S_{22}(k) &= \frac{1}{2} \sum_{j=1}^{a_{min}} \left\{ \cos\left[\lambda_{min} k + j\left(\lambda_{max} - \lambda_{min}\right)\right] + \cos\left[\lambda_{min} k - j\left(\lambda_{max} + \lambda_{min}\right)\right] \right\} \\
&= \frac{1}{2} \left\{ \frac{\cos\left[\lambda_{min} k + (a_{min} + 1) \frac{\lambda_{max} - \lambda_{min}}{2}\right] \cos\left(\frac{a_{min} \lambda_{min}}{2}\right)}{\sin\left(\frac{\lambda_{max} - \lambda_{min}}{2}\right)} \right. \\
&\quad \left. + \frac{\cos\left[\lambda_{min} k - (a_{min} + 1) \frac{\lambda_{max} + \lambda_{min}}{2}\right] \cos\left(\frac{a_{min} \lambda_{min}}{2}\right)}{\sin\left(\frac{\lambda_{max} + \lambda_{min}}{2}\right)} \right\}
\end{aligned}
$$

and by successive operations we get to

$$
S_{22}(k) = -\cos\left(\frac{\lambda_{min} a_{min}}{2}\right) \left\{ \cos\left[\lambda_{min}\left(k - \frac{a_{min}}{2}\right)\right] + \frac{\sin \lambda_{min} \sin\left[\lambda_{min}\left(k - \frac{a_{min}}{2}\right)\right]}{\cos \lambda_{min} - \cos \lambda_{max}} \right\}.
\tag{24}
$$

Now, analogously to (20)

$$
S_{23}(k) = -\frac{1}{2} \left\{ \frac{\cos \lambda_{min} \sin\left[\lambda_{min}\left(k - a_{min}\right)\right]}{\sin \lambda_{min}} + \left(k - a_{min}\right) \cos\left[\lambda_{min}\left(k - a_{min}\right)\right] \right\}.
$$

Finally,

$$
\begin{aligned}
S_{24}(k) = -\frac{1}{2} \left\{ \frac{\cos\left[\lambda_{min}\left(k - a_{min}\right) + \frac{a_{min} + 1}{2}\left(\lambda_{min} + \lambda_{max}\right)\right] \sin\left[\frac{a_{min}\left(\lambda_{min} + \lambda_{max}\right)}{2}\right]}{\sin\left(\frac{\lambda_{min} + \lambda_{max}}{2}\right)} \right. \\
\left. + \frac{\cos\left[\lambda_{min}\left(k - a_{min}\right) + \frac{a_{min} + 1}{2}\left(\lambda_{min} - \lambda_{max}\right)\right] \sin\left[\frac{a_{min}\left(\lambda_{min} - \lambda_{max}\right)}{2}\right]}{\sin\left(\frac{\lambda_{min} - \lambda_{max}}{2}\right)} \right\}
\end{aligned}
$$

and analogously to (24)

$$S_{24}(k) = \cos\left(\tfrac{\lambda_{min}a_{min}}{2}\right)\left\{\cos\left[\lambda_{min}\left(k - \tfrac{a_{min}}{2}\right)\right] + \frac{\sin\lambda_{min}\sin\left[\lambda_{min}\left(k - \tfrac{a_{min}}{2}\right)\right]}{\cos\lambda_{max} - \cos\lambda_{min}}\right\}$$

so

$$S_{22}(k) + S_{24}(k) = \frac{2\cos\left(\tfrac{a_{min}\lambda_{min}}{2}\right)\sin\lambda_{min}\sin\left[\lambda_{min}\left(k - \tfrac{a_{min}}{2}\right)\right]}{\cos\lambda_{max} - \cos\lambda_{min}}$$

and then if $a_{min} < k \le \tfrac{\tau}{2}$

$$\begin{aligned}
s(k) &= \frac{\rho^2}{\tau}\left(S_{21} + S_{22} + S_{23} + S_{24}\right) \\
&= \frac{\rho^2}{\tau}\left\{\frac{a_{max} - k}{2}\cos\left(\lambda_{min}k\right) - \frac{k - a_{min}}{2}\cos\left[\lambda_{min}\left(k - a_{min}\right)\right]\right. \\
&\quad + \cot\lambda_{min}\sin\left[\lambda_{min}\left(k - \tfrac{a_{min}}{2}\right)\right]\sin\left(\tfrac{a_{min}\lambda_{min}}{2}\right) \\
&\quad \left. + \frac{2\cos\left(\tfrac{a_{min}\lambda_{min}}{2}\right)\sin\lambda_{min}\sin\left[\lambda_{min}\left(k - \tfrac{a_{min}}{2}\right)\right]}{\cos\lambda_{max} - \cos\lambda_{min}}\right\}.
\end{aligned} \tag{25}$$

Now if $a_{min} = 1$ (23) is undetermined due to the indetermination in (22). But in this case from (18)

$$\begin{aligned}
S_{13}(k) &= \frac{1}{2}\sum_{j=1}^{1-k}\left\{\cos\left(2\pi j + \pi k\right) + \cos\left(\pi k\right)\right\} \\
&= \frac{1}{2}\sum_{j=1}^{1-k}\left[2\cos\left(\pi k\right)\right] = \left(1 - k\right)\cos\left(\pi k\right) = \left(1 - k\right)\left(-1\right)^k.
\end{aligned}$$

Also from (21) we can write

$$\begin{aligned}
S_{12}(k) &= \frac{1}{2}\left\{\frac{\cos\left[(k-1)\tfrac{\pi+\lambda_{min}}{2} + \pi\right]\sin\left[\tfrac{k(\pi-\lambda_{min})}{2}\right]}{\sin\left(\tfrac{\pi-\lambda_{min}}{2}\right)}\right. \\
&\quad \left. + \frac{\cos\left[(k-1)\tfrac{\pi-\lambda_{min}}{2} + \pi\right]\sin\left[\tfrac{k(\pi+\lambda_{min})}{2}\right]}{\sin\left(\tfrac{\pi+\lambda_{min}}{2}\right)}\right\} \\
&= \frac{1}{2\cos\left(\tfrac{\lambda_{min}}{2}\right)}\left\{-\cos\left[(k-1)\tfrac{\pi+\lambda_{min}}{2}\right]\sin\left[\tfrac{k(\pi-\lambda_{min})}{2}\right]\right. \\
&\quad \left. - \cos\left[(k-1)\tfrac{\pi-\lambda_{min}}{2}\right]\sin\left[\tfrac{k(\pi+\lambda_{min})}{2}\right]\right\}
\end{aligned}$$

and by successive operations we get to

$$S_{12}(k) = \frac{(-1)^k}{2} - \frac{\cos\left[\lambda_{min}\left(k - \frac{1}{2}\right)\right]}{2\cos\left(\frac{\lambda_{min}}{2}\right)}$$

and analogously

$$S_{14}(k) = \frac{1}{2}\left\{\frac{\cos\left[(k-1)\frac{\pi+\lambda_{min}}{2} + \lambda_{min}\right]\sin\left[\frac{k(\lambda_{min}-\pi)}{2}\right]}{\sin\left(\frac{\lambda_{min}-\pi}{2}\right)}\right.$$
$$\left. + \frac{\cos\left[(k-1)\frac{\lambda_{min}-\pi}{2} + \lambda_{min}\right]\sin\left[\frac{k(\pi+\lambda_{min})}{2}\right]}{\sin\left(\frac{\pi+\lambda_{min}}{2}\right)}\right\}$$

so

$$S_{14}(k) = \frac{(-1)^{k+1}}{2} + \frac{\cos\left[\lambda_{min}\left(k + \frac{1}{2}\right)\right]}{2\cos\left(\frac{\lambda_{min}}{2}\right)}$$

and then

$$S_{12}(k) + S_{14}(k) = \frac{1}{2\cos\left(\frac{\lambda_{min}}{2}\right)}\left\{\cos\left[\lambda_{min}\left(k + \frac{1}{2}\right)\right] - \cos\left[\lambda_{min}\left(k - \frac{1}{2}\right)\right]\right\}$$
$$= -\sin\left(\lambda_{min}k\right)\tan\left(\frac{\lambda_{min}}{2}\right)$$

so finally if $k \leq a_{min} = 1$

$$s(k) = \frac{\rho^2}{\tau}\left\{\frac{1}{2}\left[\frac{-\cos\lambda_{min}\sin\left(\lambda_{min}k\right)}{\sin\lambda_{min}} + (a_{max} - k)\cos\left(\lambda_{min}k\right)\right]\right.$$
$$\left. -\sin\left(\lambda_{min}k\right)\tan\left(\frac{\lambda_{min}}{2}\right) + (1-k)(-1)^k\right\}.$$

In addition if $\lambda^- = \lambda^+$ then (23) and (25) are both undetermined, but in this case

$$s(k) = \frac{\rho^2}{\tau}\sum_{t=1}^{\tau}\cos\left(\lambda^{asy}t\right)\cos\left[\lambda^{asy}\left(t+k\right)\right]$$
$$= \frac{\rho^2}{2\tau}\sum_{t=1}^{\tau}\left\{\cos\left[\lambda^{asy}\left(2t+k\right)\right] + \cos\left(\lambda^{asy}k\right)\right\}$$
$$= \frac{\rho^2}{2\tau}\left\{\tau\cos\left(\lambda^{asy}k\right) + \underbrace{\sum_{t=1}^{\tau}\cos\left[\lambda^{asy}\left(2t+k\right)\right]}_{S_4}\right\}.$$

Now by 1.341.3 in GR $S_4 = 0$ and then

$$s(k) = \frac{\rho^2\cos\left(\lambda^{asy}k\right)}{2} \qquad \forall k.$$

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A.3 Proof of Proposition 3

Let x_t satisfy (1). Then from (5)

$$A = \sum_{t=1}^{\tau} x_t \cos(\lambda_j t) = \underbrace{\sum_{t=1}^{a^- - \varphi} \cos\left[\lambda^-(t+\varphi)\right]\cos(\lambda_j t)}_{A1}$$

$$- \underbrace{\sum_{t=a^- - \varphi + 1}^{\tau - \varphi} \cos\left[\lambda^+\left(t - a^- + \varphi\right)\right]\cos(\lambda_j t)}_{A2}$$

$$+ \underbrace{\sum_{t=\tau - \varphi + 1}^{\tau} \cos\left[\lambda^-(t - \tau + \varphi)\right]\cos(\lambda_j t)}_{A3}$$

Now

$$A1 = \sum_{t=1}^{a^- - \varphi} \cos\left[\lambda^-(t+\varphi)\right]\cos(\lambda_j t)$$

$$= \frac{1}{2}\left\{\sum_{t=1}^{a^- - \varphi} \cos\left[\left(\lambda^- + \lambda_j\right)t + \lambda^-\varphi\right] + \sum_{t=1}^{a^- - \varphi} \cos\left[\left(\lambda^- - \lambda_j\right)t + \lambda^-\varphi\right]\right\}$$

$$\tag{26}$$

and by Eq. 1.341.3 in GR we get

$$A1 = \frac{1}{2}\left\{\frac{\cos\left[\lambda^-\varphi + \frac{a^- - \varphi + 1}{2}\left(\lambda^- + \lambda_j\right)\right]\sin\left[\frac{\left(a^- - \varphi\right)\left(\lambda^- + \lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^- + \lambda_j}{2}\right)}\right.$$

$$\left.+ \frac{\cos\left[\lambda^-\varphi + \frac{a^- - \varphi + 1}{2}\left(\lambda^- - \lambda_j\right)\right]\sin\left[\frac{\left(a^- - \varphi\right)\left(\lambda^- - \lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^- - \lambda_j}{2}\right)}\right\}$$

$$\tag{27}$$

$$= \frac{1}{2}\left\{\frac{-\sin\left[\frac{\lambda^- + \lambda_j}{2} + \lambda_j\left(a^- - \varphi\right)\right] - \sin\left(\frac{\lambda^- + \lambda_j}{2} + \lambda^-\varphi\right)}{2\sin\left(\frac{\lambda^- + \lambda_j}{2}\right)}\right.$$

$$\left.+ \frac{-\sin\left[\frac{\lambda^- - \lambda_j}{2} - \lambda_j\left(a^- - \varphi\right)\right] - \sin\left(\frac{\lambda^- - \lambda_j}{2} + \lambda^-\varphi\right)}{2\sin\left(\frac{\lambda^- - \lambda_j}{2}\right)}\right\}.$$

Let $c^- = 2\sin\left(\frac{\lambda^- + \lambda_j}{2}\right)\sin\left(\frac{\lambda^- - \lambda_j}{2}\right) = \cos\lambda_j - \cos\lambda^-$. Then

$$A1 = \frac{1}{2c^-}\left\{\cos\left[\lambda^-\left(\varphi + 1\right)\right] + \cos\lambda^- \cos\left[\lambda_j\left(a^- - \varphi\right)\right]\right.$$
$$\left. - \cos\left[\lambda_j\left(a^- - \varphi + 1\right)\right] - \cos\lambda_j \cos\left(\lambda^-\varphi\right)\right\}. \tag{28}$$

Besides, we can see

$$A2 = \frac{1}{2}\left\{\sum_{t=a^- - \varphi + 1}^{\tau - \varphi}\cos\left[\lambda^+\left(t - a^- + \varphi\right) + \lambda_j t\right]\right.$$

$$\left. + \sum_{t=a^- - \varphi + 1}^{\tau - \varphi}\cos\left[\lambda^+\left(t - a^- + \varphi\right) - \lambda_j t\right]\right\}$$

$$= \frac{1}{2}\left\{\sum_{t=0}^{a^+ - 1}\cos\left[\left(\lambda^+ + \lambda_j\right)\left(t + 1\right) + \lambda_j\left(a^- - \varphi\right)\right]\right. \tag{29}$$

$$\left. + \sum_{t=0}^{a^+ - 1}\cos\left[\left(\lambda^+ - \lambda_j\right)\left(t + 1\right) - \lambda_j\left(a^- - \varphi\right)\right]\right\}$$

and proceeding as before we get

$$A2 = \frac{1}{2}\left\{\frac{\cos\left[\frac{a^+ + 1}{2}\left(\lambda^+ + \lambda_j\right) + \lambda_j\left(a^- - \varphi\right)\right]\sin\left[\frac{a^+\left(\lambda^+ + \lambda_j\right)}{2}\right]}{\sin\left[\frac{\lambda^+ + \lambda_j}{2}\right]}\right.$$

$$\left. + \frac{\cos\left[\frac{a^+ + 1}{2}\left(\lambda^+ - \lambda_j\right) - \lambda_j\left(a^- - \varphi\right)\right]\sin\left[\frac{a^+\left(\lambda^+ - \lambda_j\right)}{2}\right]}{\sin\left[\frac{\lambda^+ - \lambda_j}{2}\right]}\right\} \tag{30}$$

$$= \frac{1}{2}\left\{-\frac{\sin\left[\frac{\lambda^+ + \lambda_j}{2} - \lambda_j\left(a^+ + \varphi\right)\right] + \sin\left[\frac{\lambda^+ + \lambda_j}{2} - \lambda_j\varphi\right]}{2\sin\left(\frac{\lambda^+ + \lambda_j}{2}\right)}\right.$$

$$\left. - \frac{\sin\left[\frac{\lambda^+ - \lambda_j}{2} + \lambda_j\left(a^+ + \varphi\right)\right] + \sin\left[\frac{\lambda^+ - \lambda_j}{2} + \lambda_j\varphi\right]}{2\sin\left(\frac{\lambda^+ - \lambda_j}{2}\right)}\right\}$$

Let $c^+ = 2\sin\left(\frac{\lambda^+ + \lambda_j}{2}\right)\sin\left(\frac{\lambda^+ - \lambda_j}{2}\right) = \cos\lambda_j - \cos\lambda^+$. Then

$$A2 = -\frac{1}{2c^+}\left\{\cos\left[\lambda_j\left(\varphi - 1\right)\right] + \cos\left[\lambda_j\left(\varphi + a^+ - 1\right)\right]\right.$$
$$\left. - \cos\lambda^+ \cos\left(\lambda_j\varphi\right) - \cos\lambda^+ \cos\left[\lambda_j\left(\varphi + a^+\right)\right]\right\} \tag{31}$$

Now

$$A3 = \frac{1}{2}\left\{ \sum_{t=\tau-\varphi+1}^{\tau} \cos\left[\lambda^-\left(t-\tau+\varphi\right)+\lambda_j t\right] \right.$$
$$\left. + \sum_{t=\tau-\varphi+1}^{\tau} \cos\left[\lambda^-\left(t-\tau+\varphi\right)-\lambda_j t\right] \right\} \tag{32}$$

Then

$$A3 = \frac{1}{2}\left\{ \frac{\cos\left(\frac{\lambda^-+\lambda_j}{2}+\varphi\frac{\lambda^--\lambda_j}{2}\right)\sin\left[\frac{\varphi\left(\lambda^-+\lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^-+\lambda_j}{2}\right)} \right.$$
$$\left. + \frac{\cos\left(\frac{\lambda^--\lambda_j}{2}+\varphi\frac{\lambda^-+\lambda_j}{2}\right)\sin\left[\frac{\varphi\left(\lambda^--\lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^--\lambda_j}{2}\right)} \right\} \tag{33}$$
$$= \frac{1}{2}\left\{ \frac{\sin\left(\frac{\lambda^-+\lambda_j}{2}+\lambda^-\varphi\right)-\sin\left[\frac{\lambda^-+\lambda_j}{2}-\lambda_j\varphi\right]}{2\sin\left(\frac{\lambda^-+\lambda_j}{2}\right)} \right.$$
$$\left. + \frac{\sin\left(\frac{\lambda^--\lambda_j}{2}+\lambda^-\varphi\right)-\sin\left[\frac{\lambda^--\lambda_j}{2}+\lambda_j\varphi\right]}{2\sin\left(\frac{\lambda^--\lambda_j}{2}\right)} \right\}$$

Therefore

$$A3 = \frac{1}{2c^-}\left\{ \cos\left(\lambda_j\varphi\right)\cos\lambda^- + \cos\left(\lambda^-\varphi\right)\cos\lambda_j \right.$$
$$\left. - \cos\left[\lambda^-\left(\varphi+1\right)\right] - \cos\left[\lambda_j\left(\varphi-1\right)\right] \right\} \tag{34}$$

And finally

$$A = \frac{1}{c^+c^-}\left[\left(\cos\lambda^- - \cos\lambda^+\right)\sin\lambda_j\cos\left(\frac{a^-\lambda_j}{2}\right)\sin\left(\frac{a^-\lambda_j}{2}-\lambda_j\varphi\right)\right]$$

Express $c^+c^- = \left(\cos\lambda_j - \cos\lambda_{max}\right)\left(\cos\lambda_j - \cos\lambda_{min}\right)$. Now by Lemma 1 $\lambda_{min} < \lambda_j$ and therefore $\cos\lambda_{min} > \cos\lambda_j$ as $\lambda_j \leq \pi$. Also $\lambda_j = \lambda_{max} \Rightarrow j = \frac{\lambda_{max}+\lambda_{min}}{2\lambda_{min}} = \frac{a_{max}+a_{min}}{2a_{min}}$. Let $j^* = \frac{\lambda_{max}+\lambda_{min}}{2\lambda_{min}}$. We can distinguish the following cases:

1. $\lambda^+ \neq \lambda^-$:

   (a) If $j \neq j^* \Rightarrow c^+c^- \neq 0 \Rightarrow A \neq 0$.

(b) If $j = j^* \Rightarrow c^+ c^- = 0$. In this case $a_{max} = (2j - 1) a_{min}$ and therefore

$$\cos\left(\frac{a^-\lambda_j}{2}\right) = \cos\left(\frac{a^-\pi}{2a_{min}}\right) = \begin{cases} \cos\left[(2j-1)\frac{\pi}{2}\right] & \text{if } a^- > a^+ \\ \cos\frac{\pi}{2} & \text{if } a^- < a^+ \end{cases}$$

so $\cos\left(\frac{a^-\lambda_j}{2}\right) = 0$ and then $A = \frac{0}{0}$ undetermined. We consider here four cases:

  i. $a^- > a^+ \neq 1$: From (28) and (34)

$$\begin{aligned} A1 + A3 &= \frac{1}{2c^-}\left\{-\cos\left[\lambda^+\left(a^- - \varphi + 1\right)\right]\right. \\ &\quad + \cos\lambda^- \cos\left[\lambda^+\left(a^- - \varphi\right)\right] \\ &\quad \left. - \cos\left[\lambda^+\left(\varphi - 1\right)\right] + \cos\left(\lambda^+\varphi\right)\cos\lambda^-\right\} \\ &= 0 \end{aligned} \tag{35}$$

as $\cos\left[\lambda_j\left(a^- \pm k\right)\right] = \cos\left[\lambda_j\left(a^+ \mp k\right)\right]$. From (29)

$$A2 = \frac{1}{2}\left\{\sum_{t=0}^{a^+-1}\cos\left[2\lambda^+t + \lambda^+\left(a^- - \varphi + 2\right)\right]\right. \\ \left. + \sum_{t=0}^{a^+-1}\cos\left[-\lambda^+\left(a^- - \varphi\right)\right]\right\}$$

where the first sum is 0 by 1.341.3 in GR and then

$$A2 = \frac{a^+\cos\left[\lambda^+\left(\varphi - a^-\right)\right]}{2} = -\frac{a^+\cos\left(\lambda^+\varphi\right)}{2}$$

so

$$A = \frac{a^+\cos\left(\lambda^+\varphi\right)}{2} = \frac{a_{min}\cos\left(\lambda_{max}\varphi\right)}{2}. \tag{36}$$

  ii. $a^- > a^+ = 1$: From (35) $A1 + A3 = 0$. Now from (29)

$$\begin{aligned} A2 &= \frac{1}{2}\left\{\cos\left[2\pi + \pi\left(a^- - \varphi\right)\right] + \cos\left[-\pi\left(a^- - \varphi\right)\right]\right\} \\ &= \cos\left[\pi\left(a^- - \varphi\right)\right] = -\cos\left(\pi\varphi\right) \end{aligned} \tag{37}$$

so

$$A = \cos\left(\pi\varphi\right).$$

iii. $a^+ > a^- \neq 1$: From (26) by Eq. 1.341.3 in GR

$$A1 = \frac{1}{2} \left\{ \frac{-\cos \lambda^- \sin (\lambda^- \varphi)}{\sin \lambda^-} + \cos \left(\lambda^- \varphi \right) \left(a^- - \varphi \right) \right\}$$

and from (31)

$$A2 = -\frac{1}{2c^+} \left\{ \cos \left[\lambda^- (\varphi - 1) \right] + \cos \left[\pi - \lambda^- (\varphi - 1) \right] \right.$$
$$\left. - \cos \lambda^+ \cos \left(\lambda^- \varphi \right) - \cos \lambda^+ \cos \left(\pi - \lambda^- \varphi \right) \right\}$$
$$= 0.$$

From (32) again by Eq. 1.341.3 in GR

$$A3 = \frac{1}{2} \left\{ \frac{\cos \lambda^- \sin (\lambda^- \varphi)}{\sin \lambda^-} + \cos \left(\lambda^- \varphi \right) \varphi \right\}$$

so analogously to (36)

$$A = \frac{a^- \cos (\lambda^- \varphi)}{2} = \frac{a_{min} \cos (\lambda_{max} \varphi)}{2}.$$

iv. $a^+ > a^- = 1$: Now from (31) $A2 = 0$. And analogously to (37) from (26) and (32)

$$A1 + A3 = \cos (\pi \varphi)$$

so again
$$A = \cos (\pi \varphi).$$

2. $\lambda^+ = \lambda^- = \lambda^{asy}$. For the symmetric case we have

(a) If $j > 1 \Rightarrow \lambda_j > \lambda^{asy} \Rightarrow c^- c^+ \neq 0$, and then $A = 0$.

(b) If $j = 1 \Rightarrow \lambda_j = \lambda^{asy} \Rightarrow A = \frac{0}{0}$. But

$$A = \sum_{t=1}^{a^- - \varphi} \cos\left[\lambda^{asy}\left(t + \varphi\right)\right] \cos\left(\lambda^{asy} t\right)$$

$$+ \sum_{t=a^- - \varphi + 1}^{\tau - \varphi} \cos\left[\pi + \lambda^{asy}\left(t - \frac{\pi}{\lambda^{asy}} + \varphi\right)\right] \cos\left(\lambda^{asy} t\right)$$

$$+ \sum_{t=\tau - \varphi + 1}^{\tau} \cos\left[\lambda^{asy}\left(t - \tau + \varphi\right)\right] \cos\left(\lambda^{asy} t\right) \qquad (38)$$

$$= \sum_{t=1}^{\tau} \cos\left[\lambda^{asy}\left(t + \varphi\right)\right] \cos\left(\lambda^{asy} t\right)$$

$$= \frac{1}{2}\left\{ \underbrace{\sum_{t=1}^{\tau} \cos\left[\lambda^{asy}\left(2t + \varphi\right)\right]}_{A5} + \sum_{t=1}^{\tau} \cos\left(\lambda^{asy} \varphi\right) \right\}$$

and then again by 1.341.3 in GR $A5 = 0$ so

$$A = \frac{\tau}{2}\cos\left(\lambda^{asy}\varphi\right).$$

Focusing now on the part involving sines,

$$B = \sum_{t=1}^{\tau} x_t \sin\left(\lambda_j t\right) = \underbrace{\sum_{t=1}^{a^- - \varphi} \cos\left[\lambda^-\left(t + \varphi\right)\right] \sin\left(\lambda_j t\right)}_{B1}$$

$$- \underbrace{\sum_{t=a^- - \varphi + 1}^{\tau - \varphi} \cos\left[\lambda^+\left(t - a^- + \varphi\right)\right] \sin\left(\lambda_j t\right)}_{B2} \qquad (39)$$

$$+ \underbrace{\sum_{t=\tau - \varphi + 1}^{\tau} \cos\left[\lambda^-\left(t - \tau + \varphi\right)\right] \sin\left(\lambda_j t\right)}_{B3}$$

Proceeding as before and using equation 1.341.2 in GR we get (complete details are available from the author upon request)

$$B1 = \frac{1}{2}\left\{ \frac{\sin\left[\lambda^- \varphi + \frac{a^- - \varphi + 1}{2}\left(\lambda^- + \lambda_j\right)\right] \sin\left[\frac{\left(a^- - \varphi\right)\left(\lambda^- + \lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^- + \lambda_j}{2}\right)} \right.$$

$$\left. - \frac{\sin\left[\lambda^- \varphi + \frac{a^- - \varphi + 1}{2}\left(\lambda^- - \lambda_j\right)\right] \sin\left[\frac{\left(a^- - \varphi\right)\left(\lambda^- - \lambda_j\right)}{2}\right]}{\sin\left(\frac{\lambda^- - \lambda_j}{2}\right)} \right\} \qquad (40)$$

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so

$$B1 = -\frac{1}{2c^-} \left\{ \cos \lambda^- \sin \left[\lambda_j \left(a^- - \varphi \right) \right] \right.$$
$$\left. - \left(\cos \left(\lambda^- \varphi \right) \sin \lambda_j - \sin \left[\lambda_j \left(a^- - \varphi + 1 \right) \right] \right) \right\} \tag{41}$$

Now

$$B2 = \frac{1}{2} \left\{ \frac{\sin \left[\frac{a^+ + 1}{2} \left(\lambda^+ + \lambda_j \right) + \lambda_j \left(a^- - \varphi \right) \right] \sin \left[\frac{a^+ \left(\lambda^+ + \lambda_j \right)}{2} \right]}{\sin \left[\frac{\lambda^+ + \lambda_j}{2} \right]} \right.$$
$$\left. - \frac{\sin \left[\frac{a^+ + 1}{2} \left(\lambda^+ - \lambda_j \right) - \lambda_j \left(a^- - \varphi \right) \right] \sin \left[\frac{a^+ \left(\lambda^+ - \lambda_j \right)}{2} \right]}{\sin \left[\frac{\lambda^+ - \lambda_j}{2} \right]} \right\} \tag{42}$$

and then

$$B2 = \frac{1}{2c^+} \left\{ \sin \left[\lambda_j \left(\varphi - 1 \right) \right] + \sin \left[\lambda_j \left(a^+ + \varphi - 1 \right) \right] \right.$$
$$\left. - \cos \lambda^+ \sin \left(\lambda_j \varphi \right) - \cos \lambda^+ \sin \left[\lambda_j \left(\varphi + a^+ \right) \right] \right\} \tag{43}$$

Finally

$$B3 = \frac{1}{2} \left\{ \frac{\sin \left(\frac{\lambda^- + \lambda_j}{2} + \varphi \frac{\lambda^- - \lambda_j}{2} \right) \sin \left[\frac{\varphi \left(\lambda^- + \lambda_j \right)}{2} \right]}{\sin \left(\frac{\lambda^- + \lambda_j}{2} \right)} \right.$$
$$\left. - \frac{\sin \left(\frac{\lambda^- - \lambda_j}{2} + \varphi \frac{\lambda^- + \lambda_j}{2} \right) \sin \left[\frac{\varphi \left(\lambda^- - \lambda_j \right)}{2} \right]}{\sin \left(\frac{\lambda^- - \lambda_j}{2} \right)} \right\} \tag{44}$$

so

$$B3 = \frac{1}{2c^-} \left\{ \cos \left(\varphi \lambda^- \right) \sin \lambda_j - \cos \lambda^- \sin \left(\varphi \lambda_j \right) - \sin \left[\lambda_j \left(1 - \varphi \right) \right] \right\} \tag{45}$$

Then

$$B = -\frac{1}{c^+ c^-} \left[\left(\cos \lambda^- - \cos \lambda^+ \right) \sin \lambda_j \cos \left(\frac{a^- \lambda_j}{2} \right) \cos \left(\frac{a^- \lambda_j}{2} - \lambda_j \varphi \right) \right]$$

We distinguish again the following cases:

1. $\lambda^+ \neq \lambda^-$

(a) If $j \neq j^* \Rightarrow c^- c^+ \neq 0 \Rightarrow B \neq 0$

(b) If $j = j^* \Rightarrow c^- c^+ = 0$ and analogously as before $B = \frac{0}{0}$, an indetermination. Then, we may have the following situations:

  i. $a^- > a^+ \neq 1$. From (41) and (45)

$$B1 + B3 = \frac{1}{2c^-} \left\{ \cos \lambda^- \sin \left[\lambda^+ \left(a^- - \varphi \right) \right] - \sin \left[\lambda^+ \left(a^- - \varphi + 1 \right) \right] \right.$$
$$\left. - \cos \lambda^- \sin \left(\varphi \lambda^+ \right) - \sin \left[\lambda^+ \left(1 - \varphi \right) \right] \right\}$$
$$= 0$$

as $\sin \left[\lambda_j \left(a^- \pm k \right) \right] = - \sin \left[\lambda_j \left(a^+ \mp k \right) \right]$.
Now from (39)

$$B2 = \sum_{t=a^- - \varphi + 1}^{\tau - \varphi} \cos \left[\lambda^+ \left(t - a^- + \varphi \right) \right] \sin \left(\lambda^+ t \right)$$
$$= \frac{1}{2} \sum_{t=a^- - \varphi + 1}^{\tau - \varphi} \left\{ \sin \left[\lambda^+ \left(2t - a^- + \varphi \right) \right] - \sin \left[\lambda^+ \left(-a^- + \varphi \right) \right] \right\}$$
$$= \frac{1}{2} \left\{ \sum_{k=0}^{a^+ - 1} \sin \left[\lambda^+ \left(2k + a^- - \varphi + 2 \right) \right] - \sum_{k=0}^{a^+ - 1} \sin \left[\lambda^+ \left(-a^- + \varphi \right) \right] \right\}$$

Now by Eq. 1.341.1 in GR the first sum is 0 so

$$B2 = -\frac{a^+}{2} \sin \left[\lambda^+ \left(-a^- + \varphi \right) \right] = -\frac{a^+}{2} \sin \left[\lambda^+ \left(a^+ + \varphi \right) \right]$$
$$= -\frac{a^+}{2} \sin \left(\pi + \varphi \lambda^+ \right) = \frac{a^+}{2} \sin \left(\lambda^+ \varphi \right)$$

and then we get

$$B = -\frac{a^+ \sin \left(\lambda^+ \varphi \right)}{2} = -\frac{a_{min} \sin \left(\lambda_{max} \varphi \right)}{2}. \qquad (46)$$

  ii. $a^+ > a^- \neq 1$. We start from (39)

$$B1 = \sum_{t=1}^{a^- - \varphi} \cos \left[\lambda^- \left(t + \varphi \right) \right] \sin \left(\lambda^- t \right)$$
$$= \frac{1}{2} \sum_{t=1}^{a^- - \varphi} \left\{ \sin \left[\lambda^- \left(2t + \varphi \right) \right] - \sin \left(\lambda^- \varphi \right) \right\}$$
$$= \frac{1}{2} \left\{ \underbrace{\sum_{t=0}^{a^- - \varphi - 1} \sin \left(2\lambda^- t + \varphi \lambda^- + 2\lambda^- \right)}_{B11} - \underbrace{\sum_{t=1}^{a^- - \varphi} \sin \left(\lambda^- \varphi \right)}_{B12} \right\}$$

Now by Eq. 1.341.1 in GR

$$B11 = \frac{\sin\left(\pi + \lambda^-\right)\sin\left(\lambda^-\varphi\right)}{\sin\lambda^-} = -\sin\left(\lambda^-\varphi\right)$$

and $B12 = \left(a^- - \varphi\right)\sin\left(\lambda^-\varphi\right)$ so

$$B1 = \frac{-a^- + \varphi - 1}{2}\sin\left(\lambda^-\varphi\right)$$

From (43)

$$B2 = -\frac{1}{2c^+}\left\{\cos\left[\lambda^-\left(\varphi - 1\right)\right] + \cos\left[\pi - \lambda^-\left(\varphi - 1\right)\right]\right.$$
$$\left. - \cos\lambda^+\cos\left(\lambda^-\varphi\right) - \cos\lambda^+\cos\left(\pi - \lambda^-\varphi\right)\right\}$$
$$= 0$$

Finally from (39)

$$B3 = \sum_{t=\tau-\varphi+1}^{\tau}\cos\left[\lambda^-\left(t - \tau + \varphi\right)\right]\sin\left(\lambda^-t\right)$$
$$= \frac{1}{2}\sum_{t=\tau-\varphi+1}^{\tau}\left\{\sin\left[\lambda^-\left(2t - \tau + \varphi\right)\right] - \sin\left[\lambda^-\left(-\tau + \varphi\right)\right]\right\}$$
$$= \frac{1}{2}\left\{\underbrace{\sum_{t=0}^{\varphi-1}\sin\left[\lambda^-\left(2t + \tau - \varphi + 2\right)\right]}_{B31} - \underbrace{\sum_{t=1}^{\varphi}\sin\left[\lambda^-\left(-\tau + \varphi\right)\right]}_{B32}\right\}.$$

Then

$$B31 = \frac{\sin\left[\lambda^-\left(\tau + 1\right)\right]\sin\left(\lambda^-\varphi\right)}{\sin\lambda^-} = \sin\left(\lambda^-\varphi\right)$$

and $B32 = \varphi\sin\left(\lambda^-\varphi\right)$ so

$$B3 = \frac{1 - \varphi}{2}\sin\left(\lambda^-\varphi\right)$$

and then analogously to (46)

$$B = -\frac{a^-\sin\left(\lambda^-\varphi\right)}{2} = -\frac{a_{min}\sin\left(\lambda_{max}\varphi\right)}{2}.$$

iii. $a_{min} = 1$: Then

$$B = \sum_{t=1}^{\tau} x_t \sin(\pi t) = 0$$

2. $\lambda^+ = \lambda^- = \lambda^{asy}$

    (a) If $j > 1 \Rightarrow c^- c^+ \neq 0$, then $B = 0$.

    (b) If $j = 1 \Rightarrow \lambda_j = \lambda^{asy}$, then $c^+ c^- = 0 \Rightarrow B = \frac{0}{0}$, undetermined. But in this case by analogy with (38)

$$B = \sum_{t=1}^{\tau} \cos[\lambda^{asy}(t + \varphi)] \sin(\lambda^{asy} t)$$

$$= \frac{1}{2} \left\{ \underbrace{\sum_{t=1}^{\tau} \sin[\lambda^{asy}(2t + \varphi)]}_{B5} - \sum_{t=1}^{\tau} \sin(\lambda^{asy}\varphi) \right\}$$

and by 1.342.1 in GR $B5 = 0$ so

$$B = -\frac{\tau}{2} \sin(\lambda^{asy}\varphi)$$

Then, the periodogram ordinates verify

$$I(\lambda_j) = \frac{1}{2\pi\tau}\left(A^2 + B^2\right)$$

where A and B we have already proved that correspond to

$$A = \begin{cases} \frac{1}{c^+ c^-}\left[\left(\cos\lambda^- - \cos\lambda^+\right)\sin\lambda_j \cos\left(\frac{a^-\lambda_j}{2}\right)\sin\left(\frac{a^-\lambda_j}{2} - \lambda_j\varphi\right)\right] & \text{if } \lambda^+ \neq \lambda^- \text{ and } j \neq j^* \\ \frac{a_{min}\cos(\lambda_{max}\varphi)}{2} & \text{if } \lambda^+ \neq \lambda^- \text{ and } j = j^* \neq \frac{\tau}{2} \\ \cos(\pi\varphi) & \text{if } \lambda^+ \neq \lambda^- \text{ and } j = j^* = \frac{\tau}{2} \\ \frac{\pi}{\lambda^{asy}}\cos(\lambda^{asy}\varphi) & \text{if } \lambda^+ = \lambda^- \text{ and } j = 1 \\ 0 & \text{if } \lambda^+ = \lambda^- \text{ and } j > 1 \end{cases}$$

and

$$B = \begin{cases} -\frac{1}{c^+ c^-}\left[\left(\cos\lambda^- - \cos\lambda^+\right)\sin\lambda_j \cos\left(\frac{a^-\lambda_j}{2}\right)\cos\left(\frac{a^-\lambda_j}{2} - \lambda_j\varphi\right)\right] & \text{if } \lambda^+ \neq \lambda^- \text{ and } j \neq j^* \\ -\frac{a_{min}\sin(\lambda_{max}\varphi)}{2} & \text{if } \lambda^+ \neq \lambda^- \text{ and } j = j^* \neq \frac{\tau}{2} \\ -\frac{\pi}{\lambda^{asy}}\sin(\lambda^{asy}\varphi) & \text{if } \lambda^+ = \lambda^- \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Besides we can see that

$$\cos\left(\frac{a^-\lambda_j}{2}\right) = \cos\left(\frac{\pi j \lambda^+}{\lambda^- + \lambda^+}\right) = (-1)^j \cos\left(\frac{\pi j \lambda^-}{\lambda^- + \lambda^+}\right)$$

as $j \in \mathbb{Z}^+$, so if $\lambda^+ \neq \lambda^-$ and $j \neq j^*$ we can express $I(\lambda_j)$ as

$$I(\lambda_j) = \frac{\lambda^{asy}}{4\pi^2} \left[\frac{(\cos\lambda^- - \cos\lambda^+)\sin\lambda_j \cos\left(\dfrac{\pi j}{2j^*}\right)}{(\cos\lambda_j - \cos\lambda^-)(\cos\lambda_j - \cos\lambda^+)}\right]^2$$

Finally, considering $\varphi \in \mathbb{N}$, we get (7).