# Fair School Placement

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## Abstract

This paper introduces $\tau$-fairness as a compromise solution reconciling Pareto efficiency and equity in School Choice Problems. We show that, by considering a weak notion of equity that we refer to as $\lambda$-equity, it is possible to contribute positively to solve an open debate, originated by the efficiency-equity trade-off of the schooling problem. We also suggest a slight modification to some allocative procedures recently introduced in the United States to compute $\tau$-fair allocations and provide support for this (possible) reformulation.

**Keywords:** School Choice Problem, Fair Matching.

**JEL codes:** C78, D63, I28.

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# I. Introduction

In this paper, we propose a notion of fairness that reconciles the requirements of efficiency and equity in School Choice Problems. We call this notion $\tau$-fairness. Additionally, we provide an allocation procedure that proposes, for each School Choice Problem, an allocation satisfying $\tau$-fairness. This procedure allows any District School Board (DSB) to implement a $\tau$-fair allocation according to the stated preferences.

Every year, many municipalities face the problem of allocating the available public school places among a large number of new applicants. Municipalities entrust this task to DSBs, which are concerned with how places are distributed through the systematic application of priorities and rules.

Some DSBs resort to School Choice Systems because of the difficulties associated with providing students with places in their schools of choice. A School Choice System first determines how to prioritize seemingly identical students. Then, it decides how priorities should be combined with the preferences of the students to generate an allocation. In this paper we assume that priorities are exogenously determined, and we concentrate on the allocation procedure.

There are two main allocation procedures used by the DSBs. The first procedure is known as the Boston mechanism (BM) because it was used in the Boston area until 2005. The BM is an example of a decentralized system for School Choice Problems. It provides efficient allocations based on student preferences. The second procedure, the Student Optimal Stable mechanism (SOSM) is the primary example of a centralized system for School Choice Problems. The SOSM equitably balances the preferences of students with the priorities of the schools.

The choice of a specific procedure depends on the objective of the DSB. When efficiency is the main goal, the DSB might use the BM; in contrast, the SOSM is a useful option when the objective is to reconcile the preferences of the students with the priorities of the school in an equitable manner. Taking the priority lists as given, the classic notions of efficiency and equity might not be applicable to School Choice Problems.[1] The classical concept of fairness in the School Choice

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[1] Alcalde and Romero-Medina (2011a) explore how to prioritize students to avoid the equity-efficiency trade-off.

framework is based on the confluence of equity and efficiency. Balinski and Sön-mez (1999) proposed a reinterpretation of the stability concept used in matching markets as an equity criterion in the School Choice Problem. Following this approach, we suggest a weaker notion of equity, which we call $\lambda$-equity. This new notion will allow us to define $\tau$-fairness as the conjunction of both $\lambda$-equity and Pareto efficiency.

To define $\lambda$-equity, we follow an old tradition in (cooperative) game theory. Aumann and Maschler (1964) proposed an iterative process to argue that some instability should not be (credibly) taken into account. With this idea in mind, we proceed as follows. Any student can object to a given allocation by claiming that it fails to be equitable according to the priorities of the school. If a student objects to the initial allocation, she must propose an alternative allocation. If no student claims that the new allocation is not equitable, the initial proposal fails to meet the equity criterion. Otherwise, we dismiss the initial objection. If an allocation is objected to without a counter objection or is not objected to at all, it is $\lambda$-equitable. Note that the logic of the process is very similar to the definition of the Bargaining Set proposed by Zhou (1994).

The concept of $\tau$-fairness introduces two interesting new features to the School Choice Problem. First, each School Choice Problem has at least one $\tau$-fair allocation. Second, there are systematic procedures for selecting a $\tau$-fair allocation for each School Choice Problem. In this paper, we propose a simple mechanism that always provides a $\tau$-fair allocation for each School Choice Problem. This procedure, which we call *the Compensating Exchange Places Mechanism* (CEPM), [2] operates in two phases and works as follows:

Phase 1. First, it proposes an equitable and, thus, $\lambda$-equitable tentative allocation. This can be accomplished by using the SOSM.

Phase 2. Because the above allocation might fail to be efficient, we consider a pure exchange economy where the agents are the students and the exchangeable

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[2]Alcalde and Romero-Medina (2011b) introduce a family of matching procedures called the Exchanging Places Mechanisms, the elements of which propose a $\tau$-fair allocation. In this paper, for simplicity of exposition, we select one specific rule, the CEPM, designed to (possibly) compensate students for the *negative* effects induced by the "tie-breaking" lottery.

goods are the places. Finally, the initial endowment for each student is the place that was tentatively assigned to her in Phase 1.

Our contribution has to be understood in the context of the reforms established in the schooling systems. In fact, the game-theoretical analysis of the BM can be seen as the origin of important reforms in the admission systems used in elementary schools in some areas of the US.[3] Simultaneously to the above reforms, other changes have been suggested in different countries.[4]

This redesign effort has resulted in the adoption of the SOSM as the best option for schooling systems (i.e., for almost one-sided markets with indifferences). However, as Abdulkadiroğlu et al. (2009), Erdil and Ergin (2008), and Kesten (2010) agree, the welfare lost due to the adoption of the SOSM can be disturbingly large.

We have seen how the implementation of $\tau$-fair allocations requires minimal reforms in the SOSM and avoids its lack of efficiency while simultaneously respecting a well-defined equity criterion. In fact, related ideas have inadvertently been proposed by some authors. These authors have either used a mechanism similar to the CEPM in order to compute the welfare lost due to the use of the SOSM (see Abdulkadiroğlu et al., 2009) or proposed efficiency corrections of the SOSM (see, e.g., Kesten, 2010) such as the Efficiency Adjusted Deferred Acceptance mechanism (EADAM), which computes an allocation that we can now recognize as $\tau$-fair.

In this paper, we propose to adopt the CEPM, a procedure for selecting a $\tau$-fair allocation. The CEPM accomplishes this by introducing a pure-exchange market for places where students have property (or usufruct) rights to the places they have been assigned by the SOSM. The theory behind our proposal has already been used in many decentralized environments where individuals have endowments and there is room for improvement.

Although it is unusual to argue that students can exchange the places they have been assigned, there are similar situations in public administrations where agents

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[3]For example, see the papers by Abdulkadiroğlu et al. (2009) related to the New York City High School Match or Abdulkadiroğlu et al. (2006) related to the Boston Public School Match.

[4]For example, see the papers by Biró (2008) on the Hungarian system, Feng (2005) on some high school processes in China, Shelim and Salem (2009) on the process in Egypt, or Ting (2007) for a study of the university access system in Taiwan.

are allowed to exchange their job positions. Therefore, in our opinion, there is legal support for the introduction of a centralized exchange of places to be phased into the allocation procedure.

Finally, it is worth noting that efficient mechanisms are being either implemented (as in the BM) or proposed (as in the EADAM proposed by Kesten, 2010) in the absence of the equity consideration that underlies the CEPM. In particular, we know that the $\tau$-fair allocations are not only $\lambda$-equitable but can also be characterized as the efficient allocations that all students weakly prefer to the ones determined by the SOSM.

We conclude this section by providing an overview of the rest of the paper. First, we introduce the basic model in Section II, and we formally state the general incompatibility between equity and efficiency in the traditional framework. Section III is devoted to introducing the main contribution of this paper; we propose a formal definition of $\tau$-fairness, and in Theorem 11, we also describe the applicability of $\tau$-fair allocations to any School Choice Problem. A formal (constructive) proof for Theorem 11 is provided in Section IV, which focuses on introducing the CEPM. The set of $\tau$-fair allocations is characterized in Section V. In Section VI, we consider the strategic properties of the CEPM. Finally, our main conclusions are gathered in Section VII. For simplicity, we relegate some technical proofs to the Appendix.

## II.    The School Choice Problem

This section is devoted to introducing several formalisms related to the School Choice Problem (SCP). We consider two sets of non-empty disjoint agents to be called students and schools. The set of students has n individuals, and is denoted by $\mathcal{S} = \{s_1, \ldots, s_i, \ldots, s_n\}$. The set of schools is denoted by $\mathcal{C}$ and has m elements (i.e., $\mathcal{C} = \{c_1, \ldots, c_j, \ldots, c_m\}$).

Each school has a (fixed) number of places to be distributed among the students, which will be called its capacity. Let $q_{c_j} \geq 1$ denote the capacity of school $c_j$, and let $Q = \left(q_{c_1}, \ldots, q_{c_j}, \ldots, q_{c_m}\right)$ denote the vector summarizing the capacities of the schools. Schools are also endowed a linear ordering prioritizing the set of students. Let $\pi_{c_j} \in \mathbb{R}^n$ be the students' ordering for school $c_j$, and let $\Pi$ be

the $(m \times n)$-matrix summarizing these priorities. Formally, $\pi_{c_j}$ is described as an n-dimensional vector such that for each $k \in \{1, \ldots, n\}$, there is a unique student $s_i$ for whom $\pi_{c_j s_i} = k$; given this description, the j-th row for matrix $\Pi$ coincides with vector $\pi_{c_j}$.

Note that using our description, no school would consider a student to be inadmissible. Most DSBs impose such a restriction in the way that the schools rank their potential students. Nevertheless, our model could easily capture the possibility of a student being inadmissible: this can be accomplished simply by introducing a new variable for each school that has defined the priority level of the last admissible student.

In contrast, each student has linear preferences over the set of schools such that no student will consider two different schools as equivalent and, additionally, that no school is considered unacceptable by a student. Let $\rho_{s_i}$ denote the ranking of the schools denoting the preferences of student $s_i$,[5] and let $\Phi$ denote the $(n \times m)$-matrix summarizing these rankings. Note that our model assumes that each student considers all the schools to be admissible.[6] Nevertheless, we can also reformulate this model by assuming that each student might consider some schools to be unacceptable. The essence of this paper is the same in both frameworks.

Therefore, a SCP can be described by listing the elements above $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$. We will say that an SCP is non-scarce whenever there are sufficient places to allocate among all the students

$$\sum_{c_j \in \mathcal{C}} q_{c_j} \geq n.$$

Given a SCP, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, a solution for it is an application $\mu$ that matches students and places. Such a correspondence is called a matching. Formally,

**Definition 1** A *matching* for $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ is a correspondence $\mu$, applying $\mathcal{S} \cup \mathcal{C}$ into itself such that:

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[5]i.e., $\rho_{s_i c_j} = 3$ indicates that student $s_i$ considers $c_j$ to be her third-best school.

[6]Here, we can also invoke the legislative regulations establishing that school attendance is compulsory for children of certain ages.

1. For each $s_i$ in $\mathcal{S}$, if $\mu(s_i) \neq s_i$, then $\mu(s_i) \in \mathcal{C}$.

2. For each $c_j$ in $\mathcal{C}$, $\mu(c_j) \subseteq \mathcal{S}$ and $|\mu(c_j)| \leq q_{c_j}$.[7]

3. For each $s_i$ in $\mathcal{S}$ and any $c_j$ in $\mathcal{C}$, $\mu(s_i) = c_j$ if and only if $s_i \in \mu(c_j)$.

The central solution concept used throughout the literature is *pair-wise stability*, introduced by Gale and Shapley (1962). In the present paper, we follow the suggestion of Balinski and Sönmez (1999) to identify this stability notion as a situation where the distribution of places among students is envy-free, which is why we say that a matching is *equitable* whenever it captures this idea of envy-freeness. Under our considerations (i.e., each school is acceptable to any student and vice versa), equity is defined as follows.

**Definition 2** A matching for $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, say $\mu$, is determined to be *equitable* if there is no student-school pair $(s_i, c_j)$ such that

1. $\rho_{s_i c_j} < \rho_{s_i \mu(s_i)}$[8] and

2. $|\mu(c_j)| < q_{c_j}$ or $\pi_{c_j s_i} < \pi_{c_j s_h}$ for some $s_h \in \mu(c_j)$.

To illustrate the function of Definition 2, let us consider a matching $\mu$, and let us assume that student $s_i$ prefers to study at school $c_j$ rather than her current school $\mu(s_i)$. If $s_i$ has a priority higher than some of the actual students attending school $c_j$ or if this school still has vacancies, then student $s_i$ might claim that the allocation process has been *unfair*.

The concept of efficiency has also been analyzed in this framework. To introduce this concept appropriately, let us remember that the only role for schools is to provide educational services to students. Therefore, the natural notion of efficiency, as proposed by Balinski and Sönmez (1999) for this framework, is the Pareto efficiency from the perspective of the students.

**Definition 3** Given a School Allocation Problem, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, matching $\mu$ is *Pareto efficient* if for any other matching $\mu' \neq \mu$, there is a student, $s_i$, such that

$$\rho_{s_i \mu(s_i)} < \rho_{s_i \mu'(s_i)}.$$

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[7]Throughout this paper, $|T|$ will denote the cardinality of set $T$.

[8]Throughout this paper, we adopt the convention that $\rho_{s_i \mu(s_i)} = m + 1$ whenever $\mu(s_i) = s_i$.

Note that for any non-scarce SCP, stability and/or efficiency of a matching $\mu$ implies that for each student $s_i$, $\mu(s_i) \in \mathcal{C}$.

A matching rule, $\mathcal{M}$, is a regular procedure that assigns a matching to each SCP. Rule $\mathcal{M}$ is said to be equitable if for any given problem, it always selects an equitable allocation. Similarly, we say that a matching rule is Pareto efficient if its outcome is always Pareto efficient relative to its input. Clearly, there are stable matching rules. In fact, any of the versions of the deferred-acceptance algorithms proposed by Gale and Shapley (1962) assigns an equitable matching for the related SCP. In contrast, the *now-or-never rule* introduced by Alcalde (1996) always selects a Pareto-efficient matching when the proposals are made by the students.

We first address the possibility of designing matching rules that always select *fair allocations* (i.e., matchings that are equitable and Pareto efficient). As Proposition 4 states, reconciling the "fairness" notion with equity and Pareto efficiency might be an impossible task.

**Proposition 4** There is no matching rule that selects an equitable and Pareto-efficient allocation for each SCP.

Proposition 4 suggests the need for a new solution concept that accurately combines the notions of equity and Pareto efficiency.

### III. $\tau$-Fairness: A New Solution Concept

In this section, we propose a new solution concept for the SCP. This concept reduces the trade-off between equity and efficiency. The central idea is to restrict the statements of students that are considered "admissible" to induce inequity in an allocation.

It is important to precisely define the objections (made by a set of agents) that should be taken into account. This definition is at the essence of both the Bargaining Set introduced by Aumann and Maschler (1964) and the solution concepts that follow in this paper. The Bargaining Set is based on the idea that any agent who formulates an objection against an allocation should propose an alternative allocation fitting some properties. Then, if an agent objects to an allocation, any other

agent might formulate an objection against this *new proposal* in the same fashion. That is, any other agent might *counter object*. The $\lambda$-equity of an allocation requires that the following criteria are met:

1. No agent will object to the allocation (i.e., it is equitable), or

2. Any objection presented by an agent will be counter objected.

We capture the concept of (weak) equity by considering only objections against an allocation that cannot be counter objected to be valid. The following example illustrates this proposal.

**Example 5** Let us consider the following SCP: $\mathcal{S} = \{1, 2, 3\}$; $\mathcal{C} = \{a, b, c\}$; $Q = (1, 1, 1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Note that matching $\mu$, with $\mu(1) = a$; $\mu(2) = b$; and $\mu(3) = c$, is not equitable because student $2$ claims that she has priority over student $1$ for school $a$. Now, let us propose the following arrangement to student $2$:

> "If you are able to propose a matching that you prefer to $\mu$ and if no other student would claim that the new proposal fails to be equitable (as you did when $\mu$ was proposed), the new matching will be implemented."

Student $2$ cannot propose such a matching.

Therefore, the arguments of the Bargaining Set that are captured by $\lambda$-equity can be informally described as follows. Let us consider a matching $\mu$. Then, any student is free to claim that this allocation fails to be equitable. Her objection must be supported by an alternative matching. The new proposal will be accepted only if no student is able to show, using identical arguments, that the new matching is also inequitable.

**Definition 6** [**Fair Objection**]

Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP, and let $\mu$ be a matching for such a problem. A *fair objection* from student $s_i \in \mathcal{S}$ against $\mu$ is a pair $(s_i, \mu')$ such that

1. $\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu(s_i)}$, and

2. $|\mu(\mu'(s_i))| < q_{\mu(\mu'(s_i))}$, or $\pi_{\mu'(s_i)s_i} < \pi_{\mu'(s_i)s_h}$ for some $s_h \in \mu(\mu'(s_i))$.

**Definition 7** [**Counter Objection**]

Let $(s_i, \mu')$ be a fair objection to matching $\mu$. A *counter objection* from student $s_h$ against $(s_i, \mu')$ is a pair $(s_h, \mu'')$ that constitutes a fair objection to matching $\mu'$.

We say that $(s_i, \mu')$ is a *justified, fair objection* to $\mu$ if it cannot be counter objected.

**Definition 8** [$\lambda$-**Equity**]

Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP. We say that matching $\mu$ is $\lambda$-*equitable* if there is no $(s_i, \mu')$ constituting a justified, fair objection to $\mu$.

Therefore, the idea of $\lambda$-equity for matching $\mu$ is that whenever a student can claim that this matching is unfair, she must be unable to propose an alternative matching that no student would consider unfair.

Note that for any given SCP, the set of $\lambda$-equitable matchings is a super-set of the sets of equitable allocations. Therefore, the following statement applies.

**Proposition 9** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP. Then it has a $\lambda$-equitable matching.

In general, there are SCPs with $\lambda$-equitable matchings that are not equitable. Notice that the matching $\mu$, proposed in Example 5, is not equitable, but it is $\lambda$-equitable.

The central solution concept that we propose in this section, $\tau$-fairness, combines two solution ideas, namely Pareto efficiency and $\lambda$-equity.

**Definition 10** [$\tau$-**Fairness**]

Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP. We say that matching $\mu$ is $\tau$-*fair* if it is Pareto efficient and $\lambda$-equitable.

The next question that we address is the existence of $\tau$-fair allocations. Although the sets of equitable and Pareto-efficient matchings might not intersect (Proposition 4), when we focus on $\lambda$-equitable allocations rather than equitable allocations, such an intersection is always non-empty.

**Theorem 11** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a School Allocation Problem. Then, it has a matching $\mu$ that is $\tau$-fair.

## IV. From Equity to $\tau$-Fairness: The CEPM

This section introduces the CEPM, a matching rule that always selects a $\tau$-fair allocation.

A simple way to define the CEPM is by the (sequential) composition of two well-known allocation procedures. The first, the input of which is a SCP, is the SOSM, which is the realization of the classic algorithm of students proposing deferred acceptance (see Gale and Shapley, 1962). When the SOSM is applied, we can interpret its outcome as a Housing Market (see Shapley and Scarf, 1974) whose agents are the students, and each individual is initially endowed with the place that the SOSM assigned to her. We can then apply Gale's Tops Trading Cycle, introduced by Shapley and Scarf (1974), to reach a Pareto improvement related to the initial outcome of the SOSM. We show in Theorem 16 that the outcome for this iterative procedure is always a $\tau$-fair matching.

Therefore, the CEPM can be seen as a constructive proof of Theorem 11, or, alternatively, as a suggestion for how to improve the system adopted by the Boston School Committee in 2005 by guaranteeing assignment efficiency.

We now provide a formal definition of the CEPM. First, Definition 12 describes how to compute, for each SCP, its Student Optimal Stable matching, $\mu^{SO}$. Second, for a given problem, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ and matching $\mu$, Definition 13 describes its *Compensating Placing Market*. Then, for a given *placing market*, Definition 14 describes how the *Tops Trading Cycle* works. We devote Appendix A to an example illustrating how to compute the CEPM for a specific SCP.

**Definition 12** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP, we define its *Student Optimal Stable Matching*, $\mu^{SO}$, as the solution of the following algorithm.

Step 1. Each student, $s_i$, applies to the school that is ranked first after $\rho_i$. Each school, $c_j$, tentatively accepts up to $q_{c_j}$ students, according its priority list, $\pi_{c_j}$. The remaining applications (if any) are rejected.

$$\bullet\ \bullet\ \bullet$$

Step k. Each student, $s_i$, applies to the first school after $\rho_i$ (if any) that has not previously rejected her application. If such a school does not exist, $s_i$ will remain unassigned. Each school, $c_j$, tentatively accepts up to $q_{c_j}$ students, according its priority list, $\pi_{c_j}$. The remaining applications (if any) are rejected.

The algorithm ends when each student who remains unassigned has been rejected by all the schools. Each student is assigned to the school (if any) that accepted her application at the last step.

The Housing Market introduced by Shapley and Scarf (1974) involves a set of agents who each own one indivisible object (her house). Each agent exhibits preferences over the houses that can be described by a linear preorder. In this model, no agent considers two distinct houses as equivalent. Following this structure, we will build, from any SCP, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, and matching $\mu$, a problem that shares the structure of the Housing Markets as described above. In this transition from a SCP and an allocation to a problem reflecting the structure of a Housing Market, there are some specifications that are (or can be seen as) natural.

In this *placing market*, a student, $s_i$, reveals that she wants to trade with $s_h$ whenever her preferences for exchange satisfy $s_h P_{s_i} s_i$. Therefore, the following can be uncontroversially assumed.

(a) Student $s_i$ wants to exchange her place with $s_h$ only if she will garner a positive benefit from such an exchange,

$$s_h P_{s_i} s_i \implies \rho_{s_i \mu(s_h)} < \rho_{s_i \mu(s_i)}.$$

(b) For any two students, $s_h$ and $s_k$, who have been assigned to different schools, any other student, $s_i$, prefers to exchange with the student who has a place-

ment in her preferred school,

$$\rho_{s_i\mu(s_h)} < \rho_{s_i\mu(s_k)} \implies s_h P_{s_i} s_k.$$

Nevertheless, there is no a priori reason justifying a student's prioritization of two different students who have been assigned a placement in the same school. A compensating placing market will consider that each student is willing to exchange with an individual who is prioritized lower in that school. The rationale of such a hypothesis is derived from the manner in which priorities are established in real-life situations. When developing a priority list, schools divide students into four categories based on two main factors (siblings and residence). With this method of classifying students, multiple students are prioritized equally by the school at this stage. Lotteries are used to break these ties. Therefore, in most cases, it is expected that the school prioritizes one student relative to another because of some random factor.[9] The proposal of the Compensating Placing Market to reverse the priority lists is intended to compensate for the random effect introduced by the lottery.[10]

**Definition 13** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP and $\mu$ a matching for this problem. We define its associated *Compensating Placing Market*, $CPM(\mu)$, as the pair $\left(\hat{S}, P\right)$ where $\hat{S}$, the set of agents, coincides with the set of students that, according to $\mu$, has been assigned a placement in some school,

$$\hat{S} = \{s_i \in \mathcal{S} : \mu(s_i) \in \mathcal{C}\}, \text{ and}$$

$P = (P_{s_i})_{s_i \in \hat{S}}$, the *preferences profile for exchange*, satisfies the condition that for each $s_i \in \hat{S}$, $P_{s_i}$ is a linear preorder on $\hat{S}$ fulfilling the following:

(a) For each $s_h \in \hat{S}$ such that $\rho_{s_i\mu(s_i)} \leq \rho_{s_i\mu(s_h)}$, $s_i P_{s_i} s_h$;

(b) For each $s_h, s_k \in \hat{S}$ such that $\rho_{s_i\mu(s_h)} < \rho_{s_i\mu(s_k)}$, $s_h P_{s_i} s_k$; and

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[9]Most systems consider several factors to categorize students. Then, a random lottery is used to break ties.

[10]Alcalde and Romero-Medina (2011b) explore a family of *placing markets*, including the CPM, each of which yields a $\tau$-fair allocation. In the present version and for the sake of simplicity, we concentrate on the CPM.

(c) For any two students $s_h$ and $s_k$ such that $\mu(s_h) = \mu(s_k) \neq \mu(s_i)$, $s_h P_{s_i} s_k$ if and only if $\pi_{\mu(s_h)s_k} < \pi_{\mu(s_h)s_h}$.

In a more general context, a *Placing Market* is a pair $(S, P)$, where $S$ is a set of agents and $P$ denotes the profile of their *preferences for exchange* (i.e., for each $s_i$ in $S$, $P_{s_i}$ is a linear preorder on $S$).

**Definition 14** We define the *Tops Trading Cycle* rule as the procedure for assigning to each *Placing Market*, $(S, P)$, the outcome for the following algorithm.

Step 1. Let us consider the digraph whose set of nodes coincides with $S$, and for any two (possibly equal) students, $s_i$ and $s_h$, there is an arc from $s_i$ to $s_h$ if $s_h$ is the maximal for $P_{s_i}$ in $S$. This digraph has at least one cycle. Let $K(S)$ be the set of students belonging to some cycle. Then, match each student in $K(S)$ to her most preferred "mate for exchanging" (i.e., if $s_i \in K(S)$, then $TTC_{s_i}(S, P) = s_h$ whenever $s_h$ is the maximal for $P_{s_i}$ in $S$).

Let us define $S^2 = S \setminus K(S)$. If $S^2$ is empty, the algorithm stops. Otherwise, go to Step 2.

$\bullet\ \bullet\ \bullet$

Step t. Let us consider the digraph whose set of nodes coincides with $S^t$, and for any two (possibly equal) students, $s_i$ and $s_h$, there is an arc from $s_i$ to $s_h$ if $s_h$ is the maximal for $P_{s_i}$ in $S^t$. This digraph has at least one cycle. Let $K(S^t)$ be the set of students belonging to some cycle. Then, match each student in $K(S^t)$ to her most preferred "mate for exchanging" (i.e., if $s_i \in K(S^t)$, then $TTC_{s_i}(S, P) = s_h$ whenever $s_h$ is the maximal for $P_{s_i}$ in $S^t$).

Let us define $S^{t+1} = S^t \setminus K(S^t)$. If $S^{t+1}$ is empty, the algorithm stops. Otherwise, go to Step $(t + 1)$.

Because there is a finite number of students, and for each $t$, it holds that $S^{t+1} \subsetneq S^t$, this algorithm stops in a finite number of steps.

We can now provide a formal definition for the *Compensating Exchange Places Mechanism*, which can be straightforwardly introduced by the next sequential procedure

(1) Given the SCP $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, let us compute its Student Optimal Stable matching, $\mu^{SO}$;

(2) Let us describe the Compensating Placing Market associated with the pair formed by $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ and $\mu^{SO}$, $CPM\left(\mu^{SO}\right)$; and

(3) Let us compute the Tops Trading Cycle outcome for $CPM\left(\mu^{SO}\right)$.

Once the process above is complete, we can distinguish two groups of students: first, those students who have not obtained a placement during the first phase (i.e., $\mu^{SO}\left(s_i\right) = s_i$) and second, the remaining students. At the end of the process, the former students will remain unassigned (i.e., $\mu^{CEP}\left(s_i\right) = \mu^{SO}\left(s_i\right) = s_i$), whereas the latter students will have the possibility of improving their initial assignment, namely $\mu^{SO}\left(s_i\right)$, if they are involved in the *exchanging process* guided by the TTC rule.

**Definition 15** [**The *Compensating Exchange Places* Mechanism**]
We define the *Compensating Exchange Places Mechanism* as the matching rule that assigns to each given SCP, say $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, the matching, $\mu^{CEP}$, such that for each $s_i \in \mathcal{S}$,

$$
\mu^{CEP}\left(s_i\right) = \begin{cases} s_i & \text{if } \mu^{SO}\left(s_i\right) = s_i \\ \mu^{SO}\left(TTC_{s_i}\left(CPM\left(\mu^{SO}\right)\right)\right) & \text{otherwise} \end{cases}.
$$

where

(a) $\mu^{SO}$ is the matching obtained when applying the SOSM to $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$; and

(b) $TTC$ is the outcome for the Tops Trading Cycle rule when applied to the Compensating Placing Market, $CPM\left(\mu^{SO}\right)$, assigned to the initial SCP, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, and its Student Optimal Stable matching, $\mu^{SO}$.

We can now establish the following result: the CEPM can be applied to yield a $\tau$-fair allocation. Therefore, Theorem 11 is a direct consequence of the result below.

**Theorem 16** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP. Its *Compensating Exchange Places matching,* $\mu^{CEP}$, is a $\tau$-Fair allocation for such a problem.

A formal proof for this result can be found in the Appendix. Nevertheless, the reader might find a heuristic explanation for why our result holds useful.

First, let us note that the $\mu^{SO}$ outcome is (constrained) Pareto efficient when restricted to the set of equitable allocations; in other words, when considering a given problem, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, for every student, $s_i$, and any equitable matching,

$$\rho_{s_i \mu^{SO}(s_i)} \leq \rho_{s_i \mu(s_i)}.$$

Second, for each SCP, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, and any matching, $\mu$, the allocation, $\mu'$, obtained by exchanging the places accordingly to $TTC$,

$$\mu'(s_i) = \begin{cases} s_i & \text{if } \mu(s_i) = s_i \\ \mu(TTC_{s_i}(CPM(\mu))) & \text{otherwise} \end{cases}.$$

is Pareto efficient and obeys the fact that $\mu$ determines school placement for students.

Together, the two observations above and $\mu^{SO}$ being equitable are crucial to the determination that $\mu^{CEP}$ is $\tau$-fair.

## V. Characterizing the Set of $\tau$-Fair Allocations

In this section, we present a complete description of the $\tau$-fair assignments of places.

The 2005 reforms in the Boston School System shifted the system from an efficient, decentralized process, the so-called BM, to an equitable, decentralized allocation mechanism, the SOSM.

If we concentrate on the trade-off between (BM) efficiency and (SOSM) equity, we can select any ($\tau$-fair) intermediate procedure minimizing this trade-off. As we pointed out in the previous section, this objective can be achieved by the CEPM. Nevertheless, as Theorem 17 reports, there are other ways to reduce the gap between efficiency and equity.

**Theorem 17** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP and $\mu$ be a matching. $\mu$ is $\tau$-fair if and only if

(a)  $\mu$ is efficient, and

(b)  For each student, $s_i$,

$$\rho_{s_i \mu(s_i)} \leq \rho_{s_i \mu^{SO}(s_i)}.$$

Theorem 17 establishes that the only Pareto-efficient allocations that are $\tau$-fair are those Pareto dominating the one that is implemented by the SOSM. Note that this result yields some straightforward consequences. The first consequence, which is the aim for Corollary 18, establishes that the unique allocation (if any) that might be equitable and Pareto efficient is the Student Optimal Stable matching. The second consequence, which is reflected in Corollary 19, reports that it is not worthwhile to distinguish between allocative *fairness* and $\tau$-*fairness* when the former concept is non-empty. Therefore, the size of the set of $\tau$-fair allocations can useful for measuring the welfare loss induced by employing the SOSM.

**Corollary 18** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP, and let $\mathcal{F}(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ denote the set of its fair matchings. Then

$$\mathcal{F}(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q) \subseteq \left\{ \mu^{SO} \right\}.$$

**Corollary 19** Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP. If its Student Optimal Stable matching is Pareto efficient, then any $\tau$-fair matching is a fair allocation.

## VI.   Strategizing the CEPM

We know that for every SCP, namely $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, its outcome (under BM) is Pareto efficient and gives the students incentives to act strategically. The school in which a student obtains a placement depends not only on her true characteristics but also on the preferences she has revealed. Additionally, for each SCP, the application of the BM combined with the students' strategic behavior yields an (expected) equitable allocation that might fail to be Pareto efficient.

The use of the SOSM in the Boston area hinged largely on the realization that students will play the BM strategically, which will, at best, induce equitable allocations (see, e.g., Abdulkadiroğlu et al., 2005, Section IV). Under the SOSM, students will reveal their preferences truthfully.

The CEPM is a combination of two strategy-proof mechanisms. First, we use the SOSM; then, we apply the TTC. However, the combination of both mechanisms cannot be strategy proof due to the impossibility of finding a Pareto-efficient and strategy-proof mechanism that Pareto dominates SOSM (see, e.g., Abdulkadiroğlu et al., 2009 or Kesten, 2010).

The violation of strategy-proofness does not necessarily imply easy manipulability. However, it is not difficult to build examples of manipulation with the CEPM.

**Example 20** Let us consider the following SCP. $\mathcal{S} = \{1, 2, 3\}$; $\mathcal{C} = \{a, b, c\}$; $Q = (1, 1, 1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \text{ and } \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

The application of the student-proposed deferred acceptance algorithm, the output of which is $\mu^{SO}$ with $\mu^{SO}(1) = b$, $\mu^{SO}(2) = a$, $\mu^{SO}(3) = c$. Therefore, the CEPM yields matching $\mu^{CEP}$ with $\mu^{CEP}(1) = a$, $\mu^{CEP}(2) = b$, and $\mu^{CEP}(3) = c$.

Student 3 can misrepresent her ranking by declaring

$$\rho'_3 = (2, 1, 3).$$

If the CEPM is applied in such a case, this student is allocated a place at school $b$. Therefore, in such a case, student 3 can manipulate the CEPM.

Let us observe that student 3's manipulation is related to a (sophisticated) risky strategy. By declaring $\rho'_3$, she provokes that, at the SOSM phase, student 1 is assigned a placement at $b$. Simultaneously, she secures a placement at $a$, the best school from 1's point of view but the worst school from her perspective.

Therefore, at the trading step, student $3$ forces student $1$ to prefer to trade with her, thus preventing student $2$ from obtaining the unique placement available at $a$. Because the SOSM is strategy-proof, when this student acts strategically, she never obtains a placement in a school that is better (from her point of view) than the one she would obtain by acting sincerely. Therefore, her only opportunity to gain from misrepresenting her preferences comes from the following combination of factors

(a) The student loses at the SOSM phase. However, due to this loss, she secures a placement at a school that is desired by other students.

(b) This desirability allows the student to gain during the trading phase.

Let us observe that the success of a student's manipulation strongly depends on the information she has about the characteristics of her rivals.[11] Roth and Rothblum (1999) and Ehlers (2008) pointed out the difficulties associated with manipulation in incomplete information environments. In particular, the absence of complete information impedes the detection and exploitation of manipulation opportunities. For example, schools that are at a similar geographic distance might be almost identical from the student's point of view. Additionally, the existence of opportunities to manipulate requires not only a consensus on the best alternative but preferences that are contrary to the general perception. If we assume that the preferences of the students are strongly positively correlated, this possibility decreases.

### VII.   Concluding Remarks

A philosophical position on the trade-off between equity and Pareto efficiency is at the origin of the modification to the mechanism used in the Boston area. The solution was to adopt the SOSM, which always selects an equitable allocation (see the description provided in Abdulkadiroğlu et al., 2006). This reform precipitated a debate in school districts such as New York and San Francisco. At the origin

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[11]In terms of the difficulty associated with manipulating the system, we want to stress that this strong requirement on knowing the characteristics of others is not necessary when the BM is employed.

of this debate was the objective to clarify, from a social perspective, the criteria behind the selection of a specific allocation rule.

The solution adopted in Boston did not solve the efficiency-equity trade-off that sparked the reform process. Moreover, there is evidence that the Boston reform generated allocations that produced large welfare losses (see Abdulkadiroğlu et al. (2009), Erdil and Ergin (2008), or Kesten (2010), among others).

This paper provides an alternative approach to circumventing the efficiency-equity dilemma. Our approach relies on a reinterpretation of the classical notion of stability in the framework of matching markets. We propose an alternative way to interpret the legitimacy of an agent's claim against the equity of an allocation. The notion of $\lambda$-equity embodies this idea and can be illustrated as follows:

> When a student claims that an allocation fails to be equitable, she is asked to propose an "alternative allocation". Then, any other student is asked whether, in her opinion, this alternative is equitable. A negative answer would only be justified by proposing a third allocation, and so on. Therefore, the first claimant's opposition to the DSB's proposal might induce an everlasting chain of *objections* and *counter objections*, or, eventually, this chain will end with the first claimant's assignment of a place that she considers worse than that which she was initially assigned.

$\lambda$-equity requires that objections against the DSB's proposal that can be counter objected should be disregarded. We find that $\lambda$-equity is a weaker notion of equity and does not conflict with the objective of implementing a Pareto-efficient allocation. In fact, the combination of the ideas of $\lambda$-equity and efficiency is at the essence of $\tau$-fairness.

We have also found a simple way to obtain $\tau$-fair allocations, the procedure called CEPM. This rule has a familiar flavor and can be implemented by introducing a minimal reform in some schooling systems, such as the procedure recently adopted in the Boston area. In short, we allow students to exchange the placements that were allocated to them in the actual system. This change, combined with our redefinition of equity, allowed us to eliminate the trade-off between efficiency and equity that was intrinsic to the previous formulation of the problem.

In reference to the ability of the CEPM to implement $\tau$-fair allocations, we can say that, as far as we know, there is no DSB that already has a system in place in which students are allowed to exchange their places. Nevertheless, there is evidence that such a reform might be legally feasible. There are related situations where agents are allowed to improve their initial allocation by exchanging their rights. This evidence is abundant both in the ranges of civil servants and in the army. For instance, civil servants are allowed to exchange their placements in Spain,[12] and a similar exchange can be performed in the US Army under the so-called *Enlisted Assignment Exchanges* (SWAPS).[13] Similar systems (i.e., agents can exchange goods that they do not own, but they *retain some rights*) can be found in some socially accepted systems, such as several *international student exchange programs* or a recent *kidney exchange*[14] program. Additionally, our procedure shares some features with the proposals by the *Ecole Démocratique* to reform the actual system in the French-speaking area of Belgium.[15]

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[12]Art. 62 in the Spanish law that governs civil servants or Law 315/1964, B.O.E 15.02.1964. This regulation can be obtained from http://www.ua.es/oia/es/legisla/funcion.htm.

[13]The reader is directed to http://usmilitary.about.com/od/armyassign/a/swap for further information on this matter.

[14]Transplant services at the Ronald Reagan UCLA Medical Center provide some information via the web page http://transplants.ucla.edu/body.cfm?id=112

[15]We would like to acknowledge Estelle Cantillon for pointing out these similarities. The proposals by the Ecole Démocratique can be found in French on its web page http://www.skolo.org/spip.php?article1126&lang=fr.

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# APPENDIX

## A.   The CEPM: An Example

This appendix provides an example illustrating how to compute the CEPM.
Let us consider the following SCP. $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $\mathcal{C} = \{a, b, c, d\}$;
the capacity for each school is $2$; and the Rankings and Priorities matrices are

$$
\Phi = \begin{bmatrix}
2 & 1 & 3 & 4 \\
2 & 4 & 1 & 3 \\
3 & 2 & 1 & 4 \\
4 & 2 & 3 & 1 \\
1 & 4 & 2 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
2 & 1 & 3 & 4
\end{bmatrix} ; \text{ and } \quad
\Pi = \begin{bmatrix}
2 & 3 & 8 & 1 & 7 & 4 & 6 & 5 \\
6 & 2 & 1 & 5 & 8 & 4 & 3 & 7 \\
7 & 5 & 6 & 8 & 2 & 3 & 1 & 4 \\
8 & 5 & 3 & 4 & 1 & 2 & 7 & 6
\end{bmatrix} .
$$

### The SOSM

The application of the student-proposed deferred acceptance algorithm, the output
of which is $\mu^{SO}$, is summarized in the following table[16]

---

[16]A row in the table indicates the applications that each school receives at each step. The
students framed in a box are those whose applications are refused, whereas the remaining students
are tentatively accepted by the school.

| *Step* | $a$ | | $b$ | | $c$ | | $d$ |
|---|---|---|---|---|---|---|---|
| 1 | $\boxed{5}$, 6, 7 | | 1, 8 | | 2, 3 | | 4 |
| 2 | 6, 7 | | 1, 8 | | 2, $\boxed{3}$, 5 | | 4 |
| 3 | 6, 7 | | 1, 3, $\boxed{8}$ | | 2, 5 | | 4 |
| 4 | 6, $\boxed{7}$, 8 | | 1, 3 | | 2, 5 | | 4 |
| 5 | 6, 8 | | $\boxed{1}$, 3, 7 | | 2, 5 | | 4 |
| 6 | 1, 6, $\boxed{8}$ | | 3, 7 | | 2, 5 | | 4 |
| 7 | 1, 6 | | 3, 7 | | $\boxed{2}$, 5, 8 | | 4 |
| 8 | 1, 2, $\boxed{6}$ | | 3, 7 | | 5, 8 | | 4 |
| 9 | 1, 2 | | 3, $\boxed{6}$, 7 | | 5, 8 | | 4 |
| 10 | 1, 2 | | 3, 7 | | 5, 6, $\boxed{8}$ | | 4 |
| 11 | 1, 2 | | 3, 7 | | 5, 6 | | 4, 8 |
| $\mu^{SO}$ := | 1, 2 | | 3, 7 | | 5, 6 | | 4, 8 |

Therefore, the SOSM proposes to allocate a placement in school $a$ to students 1 and 2; students 3 and 7 are accepted by school $b$; students 5 and 6 are placed at school $c$; and, finally, students 4 and 8 will attend school $d$.

## The Compensating Placing Market

As we mention in Section IV, a Placing Market is determined by the set of students, $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and a preference profile, $P$, denoting, for each student, how she orders her "rivals" depending on an initial matching, $\mu$. We interpret the preferences determined by a student as *her inclination to exchange* the placement that $\mu$ assigns to her with the other students. The idea of how to build these preferences is guided by the following two principles.

(a) A student participates in this market if she obtains a positive net profit from the exchange; and

(b) A student, when exchanging, wants to maximize her utility (i.e., she tries to secure a placement at the best school, according to her opinion about the educational institutions).

In the case of **CEPM**, the initial matching is $\mu^{SO}$, and a student orders two agents, other than her, who have been located at the same school by "reversing the schools ordering" proposed by the priority lists.

To illustrate how to compute the students' preferences for exchanging, $(P_{c_i})_{c_i \in \mathcal{S}}$, we will concentrate on student 6. The process is the following

(1) Because her preferred school is $a$, and $\mu^{SO}(a) = \{1, 2\}$, we note that her two "tops for exchanging" are students 1 and 2. Moreover, because $2 = \pi_{a1} < \pi_{a2} = 3$, we note that

$$2 \, P_6 \, 1.$$

(2) Now, the second school in student 6's preference list is $b$. Given that $\mu^{SO}(b) = \{3, 7\}$, we note that these students will be placed in the third and fourth positions according to $P_6$. Because $1 = \pi_{b3} < \pi_{b7} = 3$, it follows that

$$7 \, P_6 \, 3.$$

(3) Because the third school in student 6's preference list is $c$, and $\mu^{SO}(c) = \{5, 6\}$, Principle (a) above indicates that

$$6 \, P_6 \, 5.$$

For our purposes, once student 6 has established her own position on $P_6$, we do not need to continue describing how the remaining students are ordered. Nevertheless, for the sake of completeness, we will also explain how $P_6$ orders students 4 and 8.

(4) Because $\mu^{SO}(4) = \mu^{SO}(8) = d$, and $4 = \pi_{d4} < \pi_{d8} = 6$, we note that

$$8 \, P_6 \, 4.$$

Summarizing, the preferences for exchange exhibited by student 6 are

$$2 \; P_6 \; 1 \; P_6 \; 7 \; P_6 \; 3 \; P_6 \; 6 \; P_6 \; 5 \; P_6 \; 8 \; P_6 \; 4,$$

or equivalently,

$$P_6 := 2, \; 1, \; 7, \; 3, \; 6, \; 5, \; 8, \; 4.$$

Applying a similar argument to the remaining students, we can compute $P$, the description of which is

$$
\begin{aligned}
P_1 &:= \; 7, \; 3, \; 1, \; 2, \; 6, \; 5, \; 8, \; 4 \\
P_2 &:= \; 6, \; 5, \; 2, \; 1, \; 8, \; 4, \; 7, \; 3 \\
P_3 &:= \; 6, \; 5, \; 3, \; 7, \; 2, \; 1, \; 8, \; 4 \\
P_4 &:= \; 4, \; 8, \; 7, \; 3, \; 6, \; 5, \; 2, \; 1 \\
P_5 &:= \; 2, \; 1, \; 5, \; 6, \; 8, \; 4, \; 7, \; 3 \\
P_6 &:= \; 2, \; 1, \; 7, \; 3, \; 6, \; 5, \; 8, \; 4 \\
P_7 &:= \; 2, \; 1, \; 7, \; 3, \; 8, \; 4, \; 6, \; 5 \\
P_8 &:= \; 7, \; 3, \; 6, \; 5, \; 8, \; 4, \; 2, \; 1
\end{aligned}
$$

Therefore, $CPM\left(\mu^{SO}\right) = (\mathcal{S}, P)$, where $\mathcal{S} = \{1, \ldots, 8\}$ is the set of students and $P$ is the preferences profile described above.

### The Tops Trading Cycle rule

To continue with our illustrative example, we now compute the outcome for the TTC rule when applied to the Compensating Placing Market that was previously described.

Step 1. At the first step, the set of students coincides with $\mathcal{S}$. To draw the next digraph, we proceed as follows. A node is assigned to each student. We draw an arc connecting $s_i$ to $s_h$ if the latter is the top for $P_{s_i}$.[17]

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[17]For instance, because 7 is the top for $P_1$, we draw an arc that departs from 1 and is incident to 7.
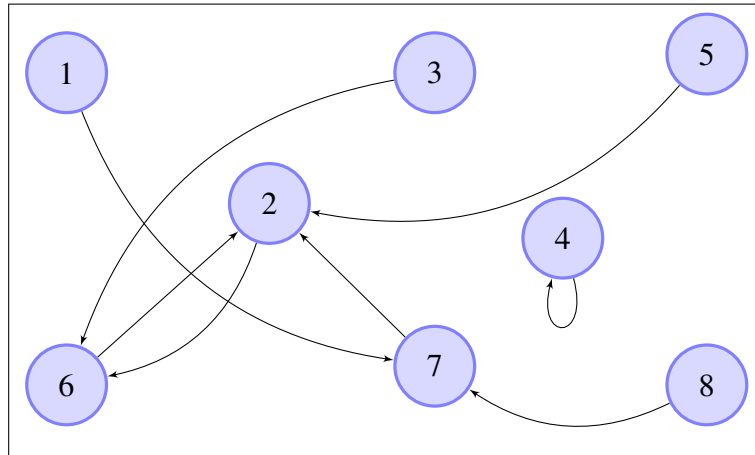
Let us observe that the above digraph contains two cycles; the first includes students $2$ and $6$, and the second contains only student $4$. Therefore, these three students exit from the market, and we can describe a "new market" whose students are $S^2 = \{1, 3, 5, 7, 8\}$

Step 2. At the second step, the set of students coincides with $S^2$ described above. We proceed in a similar manner to that explained in the previous step and draw its corresponding directed graph.
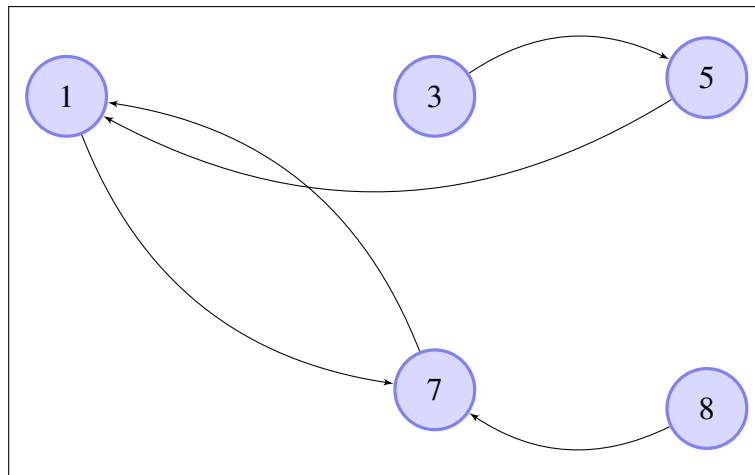


There is clearly a cycle involving students $1$ and $7$. Therefore, $K\left(S^2\right) = \{1, 7\}$, and both students exit from the market, allowing us to consider the residual market $S^3 = \{3, 5, 8\}$.

Step 3. In this step, the set of remaining students is $S^3$. It is easy to see that there are two cycles; one cycle contains student 3, whereas the other cycle only includes student 5. Therefore, $K\left(S^3\right) = \{3, 5\}$ and $S^4 = \{8\}$. To conclude,

Step 4. Because $S^4$ is a singleton, we note that $K\left(S^4\right) = \{8\}$ and that $S^5 = \emptyset$. Given that each student is, at this point, involved in some cycle, the algorithm stops.

Therefore, the outcome of the Tops Trading Cycle, applied to $CPM\left(\mu^{SO}\right)$, is

| $s_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $TTC\left(s_i\right)$ | 7 | 6 | 3 | 4 | 5 | 2 | 1 | 8 |

**The Compensating Exchange Places Mechanism**

To conclude the process, we must move from the initial matching $\mu^{SO}$ to the matching that results from the exchange of places suggested by the Tops Trading Cycle rule. For instance, because $TTC\left(1\right) = 7$, then

$$\mu^{CEP}\left(1\right) = \mu^{SO}\left(7\right) = b.$$

Therefore, the outcome of the CEPM for this example is

| $s_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\mu^{CEP}\left(s_i\right)$ | b | c | b | d | c | a | a | d |

To conclude this example and with the aim of presenting a comparison for the application of some allocation procedures relative to the data proposed in the present example, let us consider Table 1. This table assigns each student two

| | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **BM** | b | 1 | c | 1 | c | 1 | d | 1 | d | 3 | b | 1 | a | 1 | b | 1 |
| **SOSM** | a | 2 | a | 2 | b | 2 | d | 1 | c | 2 | c | 3 | b | 2 | d | 4 |
| **CEPM** | b | 1 | c | 1 | b | 2 | d | 1 | c | 2 | a | 1 | a | 1 | d | 4 |

Table 1: Comparing School Choice Mechanisms

items: the school in which she secures a placement (first item) and the position of this school in the student's ranking (second item).

Let us note that,

(1) The solution proposed by the **BM** is not $\tau$-fair. In fact, the pair $\left(5, \mu^{SO}\right)$ constitutes a *justified fair objection* to $\mu^{BM}$;

(2) When comparing the **SOSM** and the **CEPM**, it is clear that the latter Pareto dominates the former. Moreover, both mechanisms yield $\lambda$-equitable allocations.

## B.   A Proof for Proposition 4

To prove Proposition 4, let us consider the following School Allocation Problem.

$\mathcal{S} = \{1, 2, 3\}$; $\mathcal{C} = \{a, b, c\}$; $Q = (1, 1, 1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Note that there is only one stable matching in this type of problem, $\mu$, such that $\mu(1) = c$; $\mu(2) = b$; and $\mu(3) = a$. Nevertheless, $\mu$ fails to be Pareto efficient because $\mu'$, defined as $\mu'(1) = a$; $\mu'(2) = b$; and $\mu'(3) = c$, Pareto dominates $\mu$. ⊡

## C. A Proof for Theorem 17

This appendix introduces a formal proof for Theorem 17, which characterizes the set of $\tau$-fair allocations associated with each SCP. Let us remember that our result establishes that a matching, $\mu$, is $\tau$-fair if it satisfies two properties. The first property is Pareto efficiency, which is also required by Definition 10; the second is that each student (weakly) prefers the placement that $\mu$ assigns to her over the placement suggested by the SOSM,

$$\rho_{s_i \mu(s_i)} \leq \rho_{s_i \mu^{SO}(s_i)}. \tag{1}$$

**Proof of Theorem 17**

Let $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ be a SCP, and let $\mu$ be a matching. We first show that if $\mu$ is $\tau$-fair, then it satisfies Condition $(1)$. Note that by Definition 10, $\mu$ should be Pareto efficient.

To meet our objective, let us assume by way of contradiction that $\mu$ does not satisfy Condition $(1)$. There should be a student, $s_i$, such that

$$\rho_{s_i \mu^{SO}(s_i)} < \rho_{s_i \mu(s_i)}.$$

In this case, the pair $\left(s_i, \mu^{SO}\right)$ constitutes a justified, fair objection to $\mu$, which contradicts our hypothesis that $\mu$ is $\tau$-fair.

In contrast, let us assume that $\mu$ is a Pareto-efficient allocation that satisfies Condition $(1)$. We will see that $\mu$ is $\tau$-fair.

To reach a contradiction, let us assume that $\mu$ is not $\tau$-fair. Then, because $\mu$ is Pareto efficient, it should be a student-matching pair, $(s_i, \mu')$, that constitutes a justified, fair objection to $\mu$.

Therefore, by Definition 6 and Condition $(1)$, we note that

$$\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu(s_i)} \leq \rho_{s_i \mu^{SO}(s_i)}. \tag{2}$$

Note that this relationship implies that $\mu'(s_i) \in \mathcal{C}$. Let us denote such a school by $c_j$.

From Martínez et al. (2001), we know that for any matching $\hat{\mu}$, if $\rho_{s_i \hat{\mu}(s_i)} <$

$\rho_{s_i\mu^{SO}(s_i)}$ for some student, $s_i$, then $\hat{\mu}$ is not equitable. In particular, this implies that matching $\mu'$ fails to be equitable. Therefore, there should be a student, $s_h$, and a school, $c_k$, such that

(1) $\rho_{s_h c_k} < \rho_{s_k\mu'(s_h)}$, and

(2) $\pi_{c_k s_h} < \pi_{c_k s_l}$ for some $s_l \neq s_h$, or $|\mu'(c_k)| < q_{c_k}$.

Therefore, if we consider any matching, $\mu''$, such that $\mu''(s_h) = c_k$, the pair $(s_h, \mu'')$ describes a counter objection to $\mu$, which contradicts the hypothesis that $\mu$ fails to be $\tau$-fair. $\qquad\qquad\square$

## D.   A Proof for Theorem 16

To prove Theorem 16, let us consider a School Allocation Problem, $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$, and let $\mu^{SO}$ be its student optimal stable matching.

Let us observe that when describing the Compensating Places Market, the preferences for exchange held by a student, $s_i$, such that $\mu(s_i) \neq s_i$, verify that for any other student $s_h$,

$$s_h P_{s_i} s_i \Rightarrow \rho_{s_i\mu(s_h)} < \rho_{s_i\mu(s_i)}.$$

By construction, the Tops Trading Cycle satisfies the requirement that for each Placing Market, $(S, P)$, and any $s_i \in S$ such that $TTC_{s_i}(S, P) \neq s_i$,

$$TTC_{s_i}(S, P) P_{s_i} s_i.$$

This implies that for each student, $s_i$

$$\rho_{s_i\mu^{CEM}(s_i)} \leq \rho_{s_i\mu^{SO}(s_i)}. \tag{3}$$

Because $\mu^{SO}$ is equitable, we note the following.

(a) If for some student, $s_i$, $\mu^{SO}(s_i) = s_i$, then $(\mathcal{S}, \mathcal{C}, \Phi, \Pi, Q)$ is a scarce SCP,

i.e.,

$$n > \sum_{c_j \in \mathcal{C}} q_{c_j}, \text{ and}$$

(b) If there is some school, $c_j$, that has vacancies at $\mu^{SO}$, i.e.,

$$\left| \mu^{SO} (c_j) \right| < q_{c_j},$$

then for each student, $s_i$,

$$\rho_{s_i \mu^{SO}(s_i)} \leq \rho_{s_i c_j}.$$

To complete our proof, let us assume that $\mu^{CEM}$ is not $\tau$-fair. Taking into account Equation (3) and Theorem 17, we note that $\mu^{CEM}$ should not be Pareto efficient. Therefore, there should be a matching, $\mu$, such that for each student, $s_i \in \mathcal{S}$

$$\rho_{s_i \mu(s_i)} \leq \rho_{s_i \mu^{CEM}(s_i)}, \text{ and}$$

there should be a student, $s_h$, such that

$$\rho_{s_h \mu(s_h)} < \rho_{s_h \mu^{CEM}(s_h)}. \tag{4}$$

Let $S$ denote the set of students fulfilling Condition (4). By Equation (3) and given that $\mu^{SO}$ is equitable, we note that for each $s_i \in S$, there is another student, $s_h$, in $S$ such that $\mu'(s_i) = \mu^{CEM}(s_h)$. This finding implies that $s_i$'s preferences for exchange satisfy the requirement that

$$s_h P_{s_i} TTC_{s_i} \left( CPM \left( \mu^{SO} \right) \right). \tag{5}$$

To reach a contradiction, for each student $s_i \in S$, we let $t(s_i)$ denote the stage at which it is determined $TTC_{s_i} \left( CPM \left( \mu^{SO} \right) \right)$. Without loss of generality, let us assume that $s_i \in S$ is such that $t(s_i) \leq t(s_h)$ for each $s_h \in S$. By Equation (4), we note that in the digraph for stage $t(s_i)$, there is no arc from $s_i$ to $\mu^{CEM}(s_i)$.

This finding constitutes a contradiction, which points out that there is no matching, $\mu$, Pareto dominating $\mu^{CEM}$. $\quad\boxdot$