

A discusión

STRONG COMPOSITION DOWN. CHARACTERIZATIONS OF NEW AND CLASSICAL BANKRUPTCY RULES*

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ABSTRACT

This paper is devoted to the study of claims problems. We identify the family of rules that satisfy *strong composition down* (robustness with respect to reevaluations of the estate) and consistency (robustness with respect to changes in the set of agents) together. We call to that family the *backbone family*, which is a generalization of the *weighted constrained equal awards rules*. In addition, once strong composition down is combined with homogeneity only the *weighted constrained equal awards rules* survive. We also prove that the *constrained equal awards rule* is the unique rules satisfying *strong composition down* and *equal treatment of equals* together.

Keywords: strong composition down, backbone rules, constrained equal awards rule, weighted constrained equal awards rules.

JEL Classification: D63

1 Introduction

The question of the adjudication of conflicting demands was firstly addressed by O’Neill (1982). This class of claims problems refers to all those situations in which a given quantity of a certain commodity has to be distributed among some agents when the available resource falls short of the total demand. The canonical example used to illustrate such a class of problems is the allotment of the liquidation value when a firm goes bankrupt. The reader is referred to Thomson (2003) for a widely exposition of the literature.

Any *claims problem* is determined by three elements, a set of *agents*, the available amount of resource, called *estate*, and a vector of demands or *claims*. A *rule* is a way of distributing the available estate among the agents according to their claims. We follow in this work the axiomatic approach, defending the rules in terms of the properties they satisfy. Those properties usually refer to notions of equity and stability.

One of the most widely studied rules is the so-called *constrained equal awards rule*. It comes from Maimonides (12th Century), and proposes that all individuals should be treated uniformly. This implies that, for any vector of claims, all the agents that do not get her claim receive equal amounts, independently of how small or large the estate is. A generalization of this idea is behind the *weighted constrained equal awards rules*. The objective is to favor agents who are perceived as more deserving. For a given vector of positive weights, and for any vector of claims, in a *weighted constrained equal awards rule* all the agents that do not get her claim receive amounts that, when divided by their respective weights, are equal, independently of how small or large the estate is. We propose in this paper a further extension: *the backbone rules*.¹ The rules in the backbone family are in line of the two aforementioned rules. But, unlike them, for any vector of claims, all the agents that do not get her claim receive an amount that does depend on how small or large the estate is.

Among the procedural properties normally required, *composition down* emerges as a useful requirement. Imagine that when, estimating the value of the estate, we were too optimistic, and the actual value was smaller than expected. Now, two alternatives are open. Either we solve the new problem. Or we consider a problem in which the estate is the reduced one, and the claims are the allocations obtained with the overestimated estate. The property of composition down requires the final allocation to be independent of the chosen alternative. In this property is implicitly assumed that, either all agents unanimously demand the original claims, or all agents unanimously demand the awards for the overestimated estate. We propose here to revise such an assumption, and to consider the possibility that some demand their original claim while the others demand their adjusted claims to the awards for the overestimated estate. Again, two alternatives appear: either to solve the problem under the new claims vector or directly. If the final allocation is always independent of the chosen alternative, we say that our rule satisfies

¹The name will become clearer in Section 2. It is comes form its graphical representation, since it looks a backbone.

strong composition down. This new property states that agents will not benefit from insisting on their initial claims when others accept the reduction given by the tentative awards corresponding to the overestimated estate.

We also consider a property that provides robustness with respect to changes in the set of agents. *Consistency* refers to a situation in which a tentative distribution of the estate has been made, and an agent leaves the problem after accepting her award. It states the reduced problem to be solved in such a way that all remaining agents are allotted exactly the same amount as they did originally.

Our main results says that the unique solutions satisfying strong composition down and consistency together are the rules within the backbone family. Moreover, once these properties are combined with some other standard principles, such as *homogeneity* (the rules is immune to changes in the scale of the estate and claims), or *equal treatment of equals* (equal agents receive equal awards), we end up with the weighted constrained equal awards or the constrained equal awards rules, respectively.

The rest of the paper is structured as follows: In Section 2 we set up the model and we present the backbone family. In Section 3 we introduce the property of strong composition down and we present our main result. In Section 4 we explore other properties the backbone family may fulfill, as well as we provide alternative characterizations of the weighted constrained equal awards and the constrained equal awards rules. In Section 5 we conclude with some final comments and remarks. Proofs are relegated to an appendix.

2 Statement of the model. The backbone rules

Let \mathbb{N} be the set of all potential agents. Let \mathcal{N} denote the family of all finite subsets of \mathbb{N} . Let $\gamma \in \mathbb{R}$ be the upper bound, large enough, for the agents' demand.² In a claims problem, or simply a **problem**, a fixed amount $E \in \mathbb{R}_{++}$ (called **estate**) has to be distributed among a group of **agents** $N \in \mathcal{N}$ according to their **claims** (represented by $c = (c_i)_{i \in N} \in [0, \gamma]^N$) when E is not enough to fully satisfy all the claims. Therefore, a problem is given by a triple $e = (N, E, c)$ where $\sum_{i \in N} c_i \geq E$ and $c_i \leq \gamma$ for all $i \in N$. We denote by \mathbb{C}^N the class of claims problems with fixed population N , and by \mathbb{C} the class of all claims problems.

$$\mathbb{C}^N = \left\{ e = (N, E, c) \in \{N\} \times \mathbb{R}_{++} \times [0, \gamma]^N : \sum_{i \in N} c_i \geq E \right\}$$

²This upper bound is not usually considered in the literature of claims problems. All the results in this paper remain unchanged without this assumption. The only difference is the technique used to prove the main theorem. Without the upper bound it can be done by contradiction, while with this minor restriction the proof is constructive. We choose the second alternative.

and

$$\mathbb{C} = \bigcup_{N \in \mathcal{N}} \mathbb{C}^N.$$

We denote by $\mathbf{C} = \sum_{i \in N} c_i$ the **aggregate claim**.

An **awards vector** for $e \in \mathbb{C}$ is a distribution of the estate among the agents, that is, it is a list $\mathbf{x} \in \mathbb{R}_+^N$ such that: (a) Each agent receives a non-negative amount which is not larger than her claim (for each $i \in N$, $0 \leq x_i \leq c_i$); and (b) the estate is exactly distributed ($\sum_{i \in N} x_i = E$). Let $\mathbf{X}(e)$ be the set of all awards vectors for $e \in \mathbb{C}$. A **rule** is a way of selecting awards vectors, that is, it is a function, $\mathbf{S} : \mathbb{C} \rightarrow \bigcup_{e \in \mathbb{C}} \mathbf{X}(e)$, that selects, for each problem $e \in \mathbb{C}$, a unique awards vector $S(e) \in \mathbf{X}(e)$.

Let S be a rule, and let c be a fixed vector of claims, $p^S(c)$ is the path followed by $S(E, c)$ as the estate E varies from 0 to C . The path $p^S(c)$ is called **path of awards of S for c** . It is worth noting that any rule can be defined by the collection of all paths of awards for the different claims vectors.

The following are two of the most prominent rules in the literature. Each of them corresponds to different ideas of fairness in the distribution of the estate. The first one follows the Aristotelian notion of justice, and proposes a distribution of the estate proportional to the claims.

Proportional rule, p : For each $e \in \mathbb{C}$, it selects the unique awards vector $p(e) = \lambda \cdot c$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i \in N} \lambda \cdot c_i = E$.

The second comes from Maimonides (12th Century). It defends that agents should be treated equally, independently of their differences in claims. Thus, the so-called *constrained equal awards rule* proposes equality in gains, adjusting, if it is necessary, to ensure that no agent receives more than her claim.

Constrained equal awards rule, cea : For each $e \in \mathbb{C}$, it selects the unique awards vector $cea(e) = \min\{c, \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Now we consider a family of rules, the so-called *weighted constrained equal awards rules*. As its name suggests, they are a generalization of the constrained equal awards rule. In the cea rule, agents' claims are fully comparable. But it may happen that differences in agents' needs ask for some adjustments. This can be made by mean of a vector of weights. For each $i \in N$, let $\alpha_i \in \mathbb{R}_{++}$ be claimant i 's weight, and $\alpha = (\alpha_i)_{i \in N}$ the vector of weights. These weights reflect how much deserving each agent is.

Weighted constrained equal awards rule with weights $\alpha = (\alpha_i)_{i \in N}$, cea^α : For each $e \in \mathbb{C}$, it selects the unique awards vector $cea^\alpha(e) = \min\{c, \alpha\lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i \in N} \min\{c_i, \alpha_i \cdot \lambda\} = E$.

It is quite obvious that the constrained equal awards rule is a particular weighted constrained equal awards rule when all the agents have the same weight.

Figure 1 illustrates the three aforementioned rules by showing the paths of awards for several claim vectors.

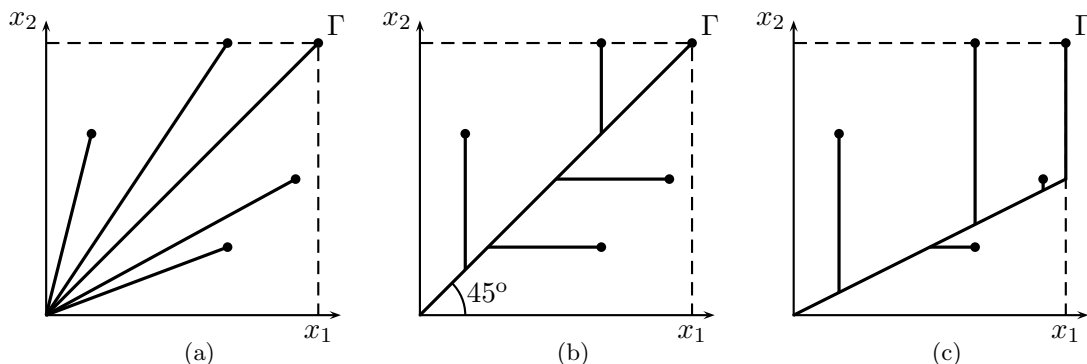


Figure 1: Path of awards for different claims in two-agents problems. (a) Proportional rule. (b) Constrained equal awards rule. (c) Weighted constrained equal awards rule for $\alpha = (2, 1)$. Γ denotes the claim vector all whose components are equal to γ , $\Gamma = (\gamma, \dots, \gamma)$.

For the sake of exposition, assume that we are in the two-agent framework, $N = \{i, j\}$. For a given rule S , and for each problem $e = (N, E, c)$, we define the **awards rate** $r^S(e)$ as the share of i th agent's award enjoyed by j when none of them is fully granted, i.e., $r^S(e) = \frac{S_j(e)}{S_i(e)}$. For the aforementioned rules, the awards rates are the following: $r^P(e) = \frac{c_j}{c_i}$, $r^{cea}(e) = 1$ and $r^{cea^\alpha}(e) = \frac{\alpha_j}{\alpha_i}$.

Therefore, the main difference, in terms of the awards rate, between the proportional and the weighted constrained equal awards rules is that, while for the first rule the awards rate depends on the claims, for the second one it is constant no matter the claims.

In comparing the constrained equal awards rule with a weighted constrained equal awards rule, both awards rates are constant independently of the agents' demand and the estate. Notice that, within the weighted constrained equal awards family, the awards rate represents how deserving agent i is in front of agent j . Here, we propose a generalization of the latter family (that we call *backbone family*), keeping the awards rate independent of the claims, but dependent on the estate. The reason for that is, even in the case that some agents are perceived as more deserving than others, such a degree of merit may vary with the size of the resource to allot. Hence, let x_i and x_j be the awards for agents i and j when none of them is fully satisfied. For the constrained equal awards rule, x_i is always equal to x_j , no matter the estate. For a weighted constrained equal awards rule, x_i is always equal to $\frac{w_i}{w_j}x_j$, no matter the estate. For the backbone rules, the

relation between x_i and x_j may depend on the estate. Hence, when the amount to divide is too little agents may receive equal or "almost" awards ($x_i = x_j$, for example), while when the the amount to divide is too large their awards may differ significantly ($x_i = x_j^2$, for example).

We provide now a formal definition of the **backbone family**. Let $p(\Gamma)$ an increasing (and continuous) path from $(0, \dots, 0)$ to $\Gamma = (\gamma, \dots, \gamma)$. Associated to each possible path $p(\Gamma)$ we define the *backbone rule* for $p = p(\Gamma)$.

Backbone rule for p , B^p : For each $e \in \mathbb{C}$, it selects the unique awards vector $B^p(e) = \min\{c, \lambda^p\}$ for some $\lambda \in p(\Gamma)$ such that $\sum_{i \in N} \min\{c_i, \lambda_i^p\} = E$.³

Roughly speaking, and in terms of paths, each member of the backbone family is described a main path (or *backbone*) from which vertical and horizontal bones start (see Figure 2). It is quite obvious that the backbone family is a generalization of the weighted constrained equal awards family.

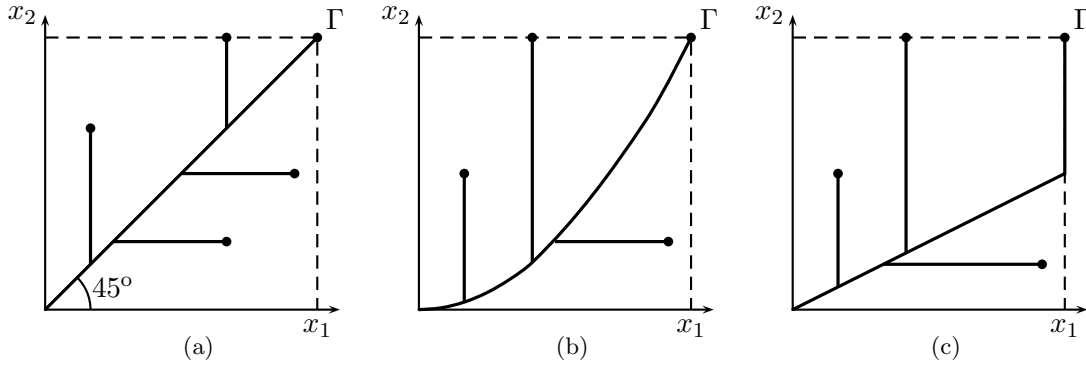


Figure 2: Illustration of the path of awards of three rules in the backbone family. (a) is the constrained equal awards rule and (c) is a weighted constrained equal awards rule. All the three rules have in common a *backbone* from $(0, \dots, 0)$ to $\Gamma = (\gamma, \dots, \gamma)$.

Example 2.1. Let $p(\Gamma)$ be a main path (backbone) that, for the two-agents case, can be parameterized by $p(\Gamma) = (z, \frac{z^2}{\gamma})$ with $z \in [0, \gamma]$ (Case b in Figure 2). Next table shows how the *cea*, *cea*^(2,1), and B^p rules apply.

³Notice that, unlike the case of the weighted and non-weighted constrained equal awards rules, in this definition λ^p denotes an n -dimensional point in $p(\Gamma)$.

c	E	Rules		
		cea	$cea^{(2,1)}$	B^p
(30, 9)	6	(3, 3)	(4, 2)	(5.67, 0.32)
(30, 9)	24	(15, 9)	(16, 8)	(20, 4)
(30, 9)	28	(19, 9)	(19, 9)	(22.80, 5.19)
(15, 40)	9	(4.5, 4.5)	(6, 3)	(8.30, 0.69)
(15, 40)	24	(12, 12)	(16, 8)	(15, 9)

3 Two properties. Strong composition down and consistency.

Next property has been widely studied in the literature of claims problems. And it is particularly useful when some uncertainty over the estate exists. Let $e = (N, E', c)$ be the problem to solve, and let x be the awards vector selected by a rule S for that problem. Imagine that when estimating the value of the estate, we were too optimistic, and the actual value E is smaller than expected $E < E'$. Now, two claims vectors arise as legitimate demands. The first one is the original claims vector c , and the second one is the promised awards x . *Composition down* requires that, independently of which demands vector we consider, we end up with the same allocation.⁴

Composition down: For each $e \in \mathbb{C}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $S(e) = S(N, E, S(N, E', c))$.

Therefore, composition down refers to the division of E from two different reference claims vectors. One in which the agents are claiming c , and another one in which the agents are claiming $x = S(N, E', c)$. It is worth noting that, in the formulation of the property it is implicitly assumed that, either all the agents unanimously demand the original claims c , or all the agents unanimously demand the promised award x . Next property allows intermediate situations, where one or several demanders may deviate from the consensus either on c or x . *Strong composition down*, in the spirit of composition down, requires that the allocation does not depend on such deviations. This property is also preventing strategic behaviors. If a mistake in the evaluation of the estate occurs, and a reevaluation of the estate is necessary, no agent $i \in N$ can manipulate her or other agents' allocation by imposing as demand c_i or x_i .

Strong composition down: For each $e \in \mathbb{C}$, each $E' \in \mathbb{R}_+$ such that $C > E' > E$, and each $T \subseteq N$, then $S(e) = S(N, E, (S_T(N, E', c), c_{N \setminus T}))$.

Strong composition down looks very demanding. But, actually, there are a lot of rules satisfying this property. Among them, the constrained and weighted constrained equal awards rules (the proportional, for example, does not). Any backbone rule also fulfils strong composition down.

Now let us consider a procedural property related to changes in the agent set. Suppose that,

⁴This property was formulated by Moulin (2000).

after solving a problem, some agents leave with their awards. The remaining agents re-valuate the new situation allocating among them the remaining estate. *Consistency* requires that each of these agents receive the amount they received before the re-valuation.⁵

Consistency: For each $e = (N, E, c) \in \mathbb{C}$, each $N' \subset N$, and for each $i \in N'$, $S_i(e) = S_i(N', \sum_{j \in N'} S_j(e), c_{N'})$.

Let $N, N' \in \mathcal{N}$ be two set of agents such that $N' \subset N$. Consistency implies that the projection of the path of awards for N onto the subspace relative to N' coincides with the path of awards for N' . Let us consider a rule S such that, when $N = \{1, 2, 3\}$ it coincides with the constrained equal awards, and when $N = \{1, 2\}$ it does with a dictatorial rule in favor of 1 (as the limit case of a weighted constrained equal awards rule). In comparing agent 1 with agent 2, this rule is quite fair for the second one when agent 3 is present, but extremely unfair when agent 3 is not. Consistency avoids this type of drawbacks. All the rules presented in the previous section, as they are defined there, are consistent. Nevertheless, if we consider a weighted constrained equal awards rule (resp. backbone rule) whose weights (resp. main backbone) depend on the particular set of agents involved in each problem, then such a rule is not consistent.

Now we present our main result. It identifies the family of rules that satisfy strong composition down and consistency together. The proof is in the appendix.

Theorem 3.1. *The backbone rules are the only rules satisfying strong composition down and consistency together.*

4 Fairness

Imagine that we change the units in which the estate and claims are measured, passing, for example, from dollars to euros. It is desirable that the division of the estate proposed by the rule is not affected by this change. *Homogeneity* requires that, if estate and claims are multiplied by the same positive amount, the same happens to the awards vector.

Homogeneity. For each $e = (N, E, c) \in \mathbb{C}$ and each $\lambda \in \mathbb{R}_+$, $S(N, \lambda E, \lambda c) = \lambda S(N, E, c)$.

The weighted constrained equal awards rules are homogenous. Regarding to the backbone family, some backbone rules satisfy homogeneity and some others do not.

Note that any weighted constrained equal awards rule is a member of the backbone family, and then it satisfies strong composition down and consistency. Next result sets that, in fact, only those rules fulfill strong composition down, consistency, and homogeneity together.

Corollary 4.1. *The weighted constrained equal awards rules are the only rules satisfying homogeneity, strong composition down, and consistency together.*

⁵This property has been widely studied in Thomson (1998).

When we deal with rationing situations, we always want to introduce a minimal requirement of fairness. *Equal treatment of equals* is very mild in this sense, and it simply requires equal agents to be treated equally. That is, agents with equals claims should receive equal awards.

Equal treatment of equals. For each $e \in \mathbb{C}$ and each $\{i, j\} \subseteq N$, if $c_i = c_j$ then $S_i(e) = S_j(e)$.

Corollary 4.2. *The constrained equal awards rule is the only rule satisfying equal treatment of equals, strong composition down, and consistency together.*

Most of the characterizations of the constrained equal awards rule provided by the existing literature use three types of properties, one of each type. First, those involving impartiality principles, as it is the case of fairness. Second, stability with respect to changes in the in estate, as it is the case of composition down or its dual. Finally, the third type of properties, and the most controversial one, refers to very particular value judgements. As an illustration, Herrero and Villar (2001) contains two examples of those properties, *conditional full compensation* and *exemption*, where agents with small claims are deliberately protected. The last corollary avoids the latter type of principles, imposing only impartiality and stability with non-manipulability.

5 Final remarks

In this work we have presented a new generalization of the weighted constrained equal awards rules, the backbone family. Both families have in common that they favor agents who are perceived as more deserving. The difference is that such a perception of deservingness is constant for the weighted constrained equal awards rule, while for the backbone rules it depends on the size of the estate. We have also introduced the property of strong composition down as a revision of composition down. Strong composition down and consistency characterize the backbone family. Moreover, by adding homogeneity to those properties we end up with the weighted constrained equal awards family. And by adding equal treatment of equals we end up with the constrained equal awards rule.

For the present paper, we have implicitly considered rules from the point of view of gains. Nevertheless, it is quite common in the literature to make the dual analysis as well. Two rules are dual is one of them allocates awards in the same way the other one allocates loses. Thus, the dual of the constrained equal awards and the weighted constrained equal awards rules are the so-called *constrained equal losses* and *weighted constrained equal losses rules*, respectively.⁶ Similarly, we may define a new family as that resulting from considering the dual of the backbone family. Analogously to the rules, the notion of duality is applied to the properties as well. Two properties are dual is whenever a rule satisfies one of the properties, the dual of such a rule satisfies the other property. Homogeneity and equal treatment of equals are self-dual (the dual property is itself), while the dual of composition down is *composition up*. Again, we may consider

⁶The reader is referred to Thomson (2003) for a formal description of both.

the dual property of strong composition down, whose flavor goes along the lines of composition up in the same way strong composition down does with respect to composition down. In view of Theorem 3.1, and using the characterization by duality result in Herrero and Villar (2001), the dual of strong composition down and consistency characterize the dual of the backbone family.

Appendix A. Proofs

This appendix is devoted to the proof of Theorem 3.1, preceded by some definitions and technical results.

Resource monotonicity stipulates that no agent is penalized as a consequence of an increase in the estate.

Resource monotonicity. For each $e = (N, E, c), e' = (N, E', c) \in \mathbb{C}$, if $E' > E$, then $S(e') \geq S(e)$.

Let us consider an awards vector with the following feature. For each two-agent subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. *Converse consistency* requires that the allocation is the one selected by the rule for the original problem.⁷

Let $c.con(e; S) \equiv \{x \in \mathbb{R}_+^N : \sum_{i \in N} x_i = E \text{ and for all } N' \subset N \text{ such that } |N'| = 2, x_{N'} = S(N', \sum_{i \in N'} x_i, c_S)\}$

Converse consistency. For each $e \in \mathbb{C}$, $c.con(e; S) \neq \phi$, and if $x \in c.con(e; S)$, then $x = S(e)$.

Lemma 5.1 (Elevator Lemma, Thomson (1998)). *If a rule S is consistent and coincides with a conversely consistent rule S' in the two agent case, then it coincides with S' in general.*

Proposition 5.1 (Chun (1999)). *Resource monotonicity and consistency together imply converse consistency.*

Proof of Theorem 3.1.

It is not difficult to check that any backbone rule satisfies *strong composition down* and *consistency*. It is also straightforward that *strong composition down* implies *composition down* and the latter *resource monotonicity*. Therefore, in application of Proposition 5.1, any backbone rule is *converse consistent*. Let S be a rule fulfilling *strong composition down* and *consistency*. Let us define the path $p = p^S(\Gamma)$ as the path of awards of rule S for claims vector Γ . We show that $S = B^p$. By the Elevator Lemma, it is enough to prove the result in the two-agents case. We assume that $N = \{i, j\}$ and $c_i \leq c_j$. For a given claims vector c , let $y \in p(\Gamma)$ be a vector such that $y_i = c_i$ and $y_j < c_j$. And let $z \in p(\Gamma)$ be a vector such that $z_j = c_j$ and $z_i > c_i$. We distinguish several cases.

Case 1. If $c \in p(\Gamma)$. Let $E \in [0, \sum_{i \in N} c_i]$. Note that, since $c \in p(\Gamma)$, $S(N, \sum_{i \in N} c_i, \Gamma) = c$ by definition of path of awards. Then $S(N, E, c) = S(N, E, S(\sum_{i \in N} c_i, \Gamma))$. By *strong composition down*, $S(N, E, S(\sum_{i \in N} c_i, \Gamma)) = S(N, E, \Gamma) = B^p(N, E, \Gamma) = B^p(N, E, c)$. Therefore, $S(N, E, c) = B^p(N, E, c)$.

⁷This property was formulated by Chun (1999).

Case 2. If $c \notin p(\Gamma)$ and $E \leq \sum_{i \in N} y_i$. By Case 1, $S(N, \sum_{i \in N} y_i, z) = B^p(N, \sum_{i \in N} y_i, z) = y$. Then, notice that $c = (y_{-n}, z_n) = (S_{-n}(\sum_{i \in N}, z), z_n)$. By *strong composition down*, $S(N, E, z) = S(N, E, (S_{-n}(N, \sum_{i \in N} y_i, z), z_n)) = S(N, E, c)$. Since $z \in p(\Gamma)$, by Case 1, we know that $S(N, E, z) = B^p(N, E, z) = B^p(N, E, c)$. Therefore, $S(N, E, c) = B^p(N, E, c)$.

Case 3. If $c \notin p(\Gamma)$ and $E > \sum_{i \in N} y_i$. Note that, on one hand, by *strong composition down*, $S(\sum_{i \in N} y_i, c) = S(\sum_{i \in N} y_i, S(E, c)) \leq S(E, c)$; and on the other hand, $S(\sum_{i \in N} y_i, c) = B^p(\sum_{i \in N} y_i, c) = y$. Then, $c \geq S(E, c) \geq y$. Therefore, $S(E, c) = B^p(E, c)$.

Therefore, S and B^p coincides in the two-agents case, and then they do so in general.

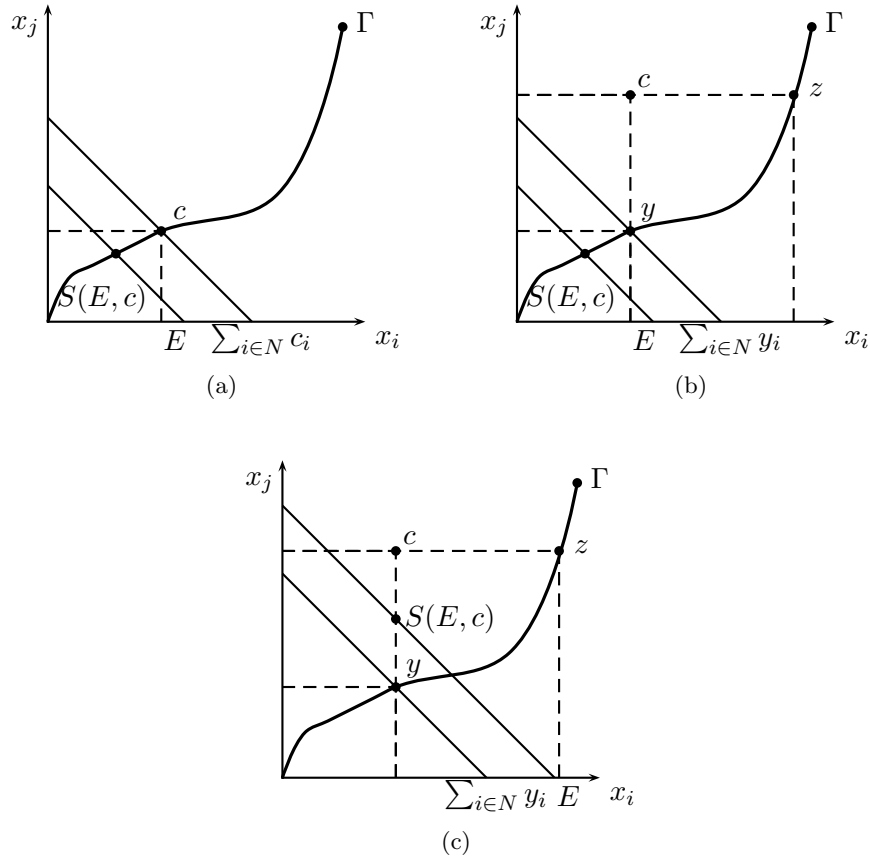


Figure 3: Illustration of the proof for the two-agent case. (a) Case 1. (b) Case 2. (c) Case 3.

Remark 5.1. The proportional rule satisfies consistency but it fails to satisfy strong composition down. A rule satisfying strong composition down but not consistency can be defined as follows.

$$S(e) = \begin{cases} cea^{(2,1)}(e) & \text{if } N = \{1, 2\} \\ cea(e) & \text{otherwise} \end{cases}$$

The properties that characterize the backbone family are, therefore, independent.

Proof of Corollary 4.2.

This proof comes from Thomson (2006). They characterize the set of homogeneous rules. By using their result it is not difficult to check that the only homogeneous rules in the backbone family are the weighted constrained equal awards rules.

Proof of Corollary 4.2.

By Theorem 3.1 is enough to show that the constrained equal awards rule is the unique backbone rule satisfying *equal treatment of equals*. In term of paths of awards, a rule S satisfies *equal treatment of equals* if and only if the diagonal is the path of awards of S for claims Γ , that is, if $(\lambda, \dots, \lambda) \in p^S(\Gamma)$ for all $\lambda \leq \gamma$. Therefore, $B^p = cea$.

References

- Chun, Y. (1999). Equivalence of axioms for bankruptcy problems. *International Journal of Game Theory*, 28:511–520.
- Herrero, C. and Villar, A. (2001). The three musketeers: four classical solutions to bankruptcy problems. *Mathematical Social Sciences*, 42:307–328.
- Moulin, H. (2000). Priority rules and other asymmetric rationing methods. *Econometrica*, 68:643–684.
- O’Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2:345–371.
- Thomson, W. (1998). Consistency and its converse: an introduction. *Rochester Center for Economics Research*, 448.
- (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, 45:249–297.
- (2006). Independence conditions for the adjudication of conflicting claims. *mimeo*.