A discusión

TULLOCK AND HIRSHLEIFER: A MEETING OF THE MINDS*

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ABSTRACT

We introduce the serial contest by building on the desirable properties of two prominent contest games. This family of contest games relies both on relative efforts (as Tullock's proposal) and on absolute effort differences (as difference-form contests). An additional desirable feature is that the serial contest is homogeneous of degree zero in contestants' efforts. The family is characterized by a parameter representing how sensitive the outcome is to contestants' efforts. It encompasses as polar cases the (fair) lottery and the (deterministic) all-pay auction. Equilibria have a close relationship to those of the (deterministic) all-pay auction and important properties of the latter hold for the serial contest, too.

Keywords: rent-seeking, (non-) deterministic contest, contest success function, all-pay auction, rent dissipation, exclusion principle, preemption effect, cap, campaign contributions.

JEL classification numbers: C72, D72, D44.

1. Introduction

In a contest game agents exert irreversible effort to increase their probability of winning a prize. Contests have been used to analyze a variety of situations including lobbying, rent-seeking and rent-defending contests, litigation, political campaigns, military conflict, patent races, arms races, sports events, promotional competition, labor market tournaments or R&D competition. Moreover, recent papers (like e.g. Alesina and Spolaore (2005), Baron and Diermeier (2006), Konrad (2000a and b) or Polborn and Klumpp (2006)) have embedded simple contest games in larger models in order to capture the effect of conflict on other variables of interest. This paper proposes a new family of contest games to model such situations.

In a contest the probability of winning is given by a contest success function (henceforth CSF) which depends on the efforts of the players. A special case is the all-pay auction, in which the player exerting the highest effort wins the prize with probability one. Such a contest is therefore called deterministic (or perfectly discriminating). The all-pay auction has been analyzed by Hillman and Riley (1989), Baye *et al.* (1993, 1996) or Che and Gale (1998), among others.[1]

The literature has frequently studied two families of CSFs. The 'classical' specification was proposed by Tullock (1980) and has been further analyzed in Pérez-Castrillo and Verdier (1992), Baye *et al.* (1994) or Skaperdas (1996). In Tullock's specification the probability of winning of bidder B_i is given by

$$\Psi_i(b) = \frac{b_i^\alpha}{\sum_{j=1}^n b_j^\alpha}, \tag{1.1}$$

where $b = (b_1, ..., b_n)$ is a vector of effort levels of the n contestants and α is a positive parameter measuring returns to scale from effort. Note that if the CSF is completely insensitive to effort ($\alpha = 0$), the extreme case of a (fair) lottery is obtained. The opposite case of extreme sensitivity ($\alpha \to \infty$) yields the all-pay auction. Tullock's formulation is also known as ratio-form, since (1.1) can be rewritten so that it depends on the ratio of contestants' efforts. It depends, hence, on a relative criterion.

Tullock's functional form has the important advantage of being homogenous of degree zero in effort. Homogeneity is a realistic property that might be interpreted as saying that it does not matter whether lobbying expenditures are measured in dollars or in euros. In addition, homogeneity is a convenient analytical property which may explain the popularity of Tullock's CSF in applications.[2]

[1] Although non-deterministic contests are also all-pay auctions, in the current paper we refer to the deterministic all-pay auction simply as all-pay auction. In keeping up with much of the existing literature, we use the terms "contestants" and "bidders" as well as "effort levels" and "bids" interchangeably.

[2] Malueg and Yates (2006, p. 719) write concerning homogeneity: "This property is intuitively appealing for rent-seeking contests. The contest winner is determined by relative efforts. For example,

The second popular contest family is called 'difference-form' CSFs. It has been proposed by Hirshleifer (1989) and further studied in Skaperdas (1996), Baik (1998) and Che and Gale (2000). Although the precise mathematical formulation of the CSF differs in these papers, the common element is that win probabilities are based on effort differences. For instance, Che and Gale propose the following piece-wise linear difference-form for contests with two bidders

$$\Psi_1(b) = max\left\{min\left\{\frac{1}{2}+\alpha(b_1-b_2),1\right\},0\right\} \text{ and } \Psi_2(b) = 1-\Psi_1(b). \tag{1.2}$$

As Tullock's specification, the positive scalar α specifies how deterministic the contest is, containing the polar cases of the (fair) lottery and the all-pay auction. Notice that, since (1.2) depends on the difference of efforts, it is not homogenous and it relies on an absolute criterion.

The allocation of the prize according to an absolute criterion is controversial. Che and Gale argue forcefully that it is appropriate in many contexts. Skaperdas (1996) and Hirshleifer (2000) point out that it is a strong property. Consider the following quote from Hirshleifer (p. 779):

> "It might be thought a fatal objection against the difference form of the CSF that a force balance of 1,000 soldiers versus 999 implies the same outcome (in terms of relative success) as 3 soldiers versus 2! That this may seem unreasonable is probably due to the exclusion of idiosyncratic and unmodelled factors that might inject a random element into the outcome. Any reasonable provision for randomness would imply a higher likelihood of the weaker side winning the 1,000:999 comparison than in the 3:2 comparison."

Summarizing, it seems that an important trade-off emerges. The choice of a CSF comparing absolute effort levels seems to imply the vulnerability against the above criticism. Moreover, one seems to be forced to give up homogeneity. In the current paper we propose a CSF reconciling these criteria. In a nutshell, we follow Hirshleifer's suggestion to weaken the absolute criterion and do this in such a way as to obtain a homogenous CSF.

Instead of postulating that win probabilities depend on the absolute mark-up b_1-b_2, we require that they depend on the percentage mark-up

$$\begin{array}{ll} \frac{b_2-b_1}{b_2} & \text{if} \quad b_1 \leq b_2 \\ \frac{b_1-b_2}{b_1} & \text{if} \quad b_1 \geq b_2 \end{array}$$

if all players double their effort, then the probabilities of winning the contest are unchanged—the increase in effort is completely wasted."

of the highest bid. So we define

$$\Psi_1(b) = \begin{cases} \frac{1}{2} - \frac{b_2 - b_1}{2b_2} & \text{if} \quad b_1 \leq b_2 \\ \frac{1}{2} + \frac{b_1 - b_2}{2b_1} & \text{if} \quad b_1 \geq b_2 \end{cases} \quad \text{and } \Psi_2(b) = 1 - \Psi_1(b). \tag{1.3}$$

Notice that this assignment process follows exactly Hirshleifer's suggestion.

The present paper proposes the serial CSF as a generalization of the previous expression.[3] In the case of two contestants the serial contest can be defined as follows

$$\Psi_1(b) = \begin{cases} \frac{1}{2}\left(\frac{b_1}{b_2}\right)^{\alpha} & \text{if} \quad b_1 \leq b_2 \\ 1 - \frac{1}{2}\left(\frac{b_2}{b_1}\right)^{\alpha} & \text{if} \quad b_1 \geq b_2 \end{cases} \quad \text{and } \Psi_2(b) = 1 - \Psi_1(b). \tag{1.4}$$

A formulation for more than two contestants will be provided in Section 2.[4] Similarly to the other two families, the family of serial contest games is characterized by a economics of scale parameter α that comprises the polar cases of the (fair) lottery and the all-pay auction. As the CSF becomes more sensitive to effort levels, the contest becomes more deterministic until the benchmark case of the all-pay auction is reached. Notice also that the serial contest is homogeneous.

The next figure compares the three families of CSFs for $\alpha \in \{1/2, 1, 10\}$. In the examples we fix the bid of the second contestant as $b_2 = 1$.[5] In the examples the piecewise linear function (red) is Che and Gale's CSF and the serial CSF (black) intersects Tullock's CSF (blue) from below. The third example shows that both the serial and Tullock's CSF can have a region with increasing marginal returns from effort. It also suggests that as the all-pay auction is approached, Tullock's CSF and the serial contest behave very similarly.

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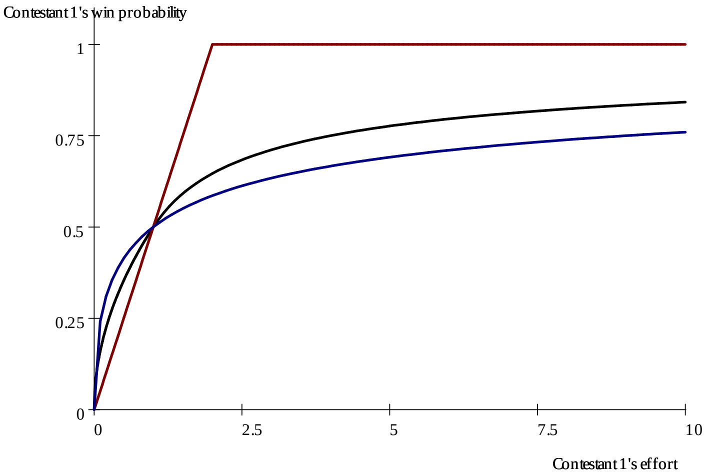
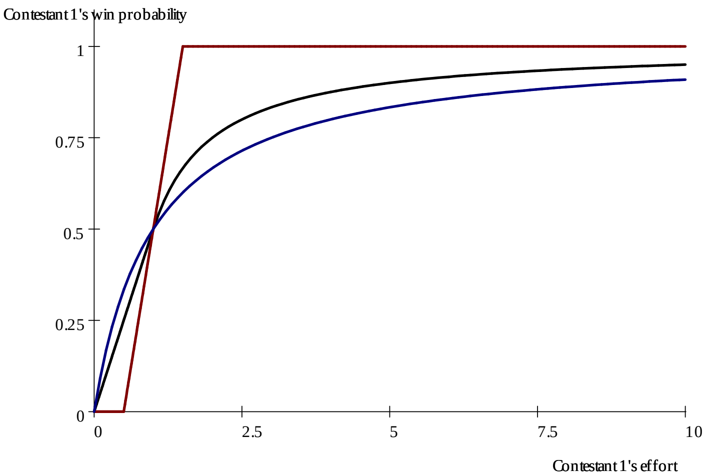
[3] The class of serial contests is related to three different literatures. First, it is inspired in a proposal for bankruptcy problems known as the Contested Garment Principle (see Dagan (1996) for an analysis of this bankruptcy solution). Second, there is a similarity to the Serial Cost Sharing Rule of Moulin and Shenker (1992). Third, there exists a close relationship to the Shapley Value of appropriately defined cooperative games (see Littlechild and Owen (1973) for the closely related airport game).

[4] Notice that for $\alpha = 1$, (1.4) boils down to (1.3).

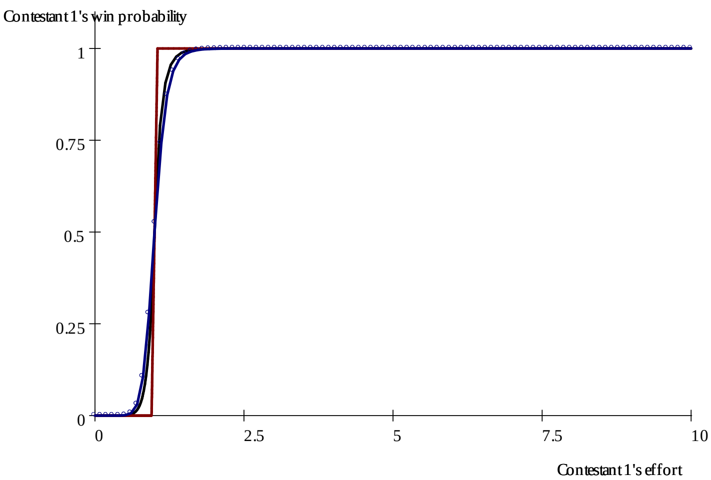
[5] Note that because of homogeneity this is w.l.o.g. under Tullock's and the serial CSF, but not with the difference-form. Under the later contestant 1's win probability is responsive to his effort when $b_1 \in [b_2 - 1/(2\alpha), b_2 + 1/(2\alpha)]$.



$\alpha = 1/2$



$\alpha = 1$



$\alpha = 10$

Figure 1.1: Three CSFs for $\alpha \in \{1/2, 1, 10\}$ and $b_2 = 1$.

The current paper offers a strategic analysis of the serial contest. For the case of two contestants we provide characterization of equilibrium for all values of the returns to scale from effort parameter α and for any valuations for the prize the contestants might have. As in the Tullock rent-seeking game, pure strategy Nash equilibria exist only if the contest is not too deterministic ($\alpha \leq 1$). For more deterministic contests we analyze mixed strategy Nash equilibria.

Surprisingly, it turns out that the equilibrium properties of the serial contest are very robust to different amounts of non-determinacy. Broadly speaking, for a wide range of non-determinacy the equilibria are essentially the ones of the extreme case of the all-pay auction ($\alpha \geq 1$). This includes the constant returns to scale case ($\alpha = 1$) in which the equilibrium is in pure strategies: *contestants bid the expectation of the mixed strategy equilibrium of the all-pay auction.* Intuitively, as the contest approaches the fair lottery further ($\alpha < 1$), the predictions of the serial contest differ more from the all-pay auction and converge to the optimal behavior in a lottery completely insensitive to effort. We investigate then to what extent these results are robust to an increase in the number of contestants. As long as the CSF is not too insensitive to effort, further contestants have a strict incentive not to participate in the contest. We show also that the close relationship between the serial contest and the all-pay auction extends to further equilibrium properties like rent dissipation, exclusion principle, preemption effect or the consequences of a cap on individual effort levels.

Summarizing, the contest proposed in the current paper has several advantages over the two previously mentioned contest families:

With respect to Tullock's formulation, we provide a CSF in which – in addition to relative efforts – absolute effort differences play a role. Whether this is appropriate depends on the context. From a more applied point of view the serial contest with two contestants allows characterization of equilibrium for all levels of sensitivity of the CSF to contestants efforts and for any valuations for the prize the contestants might have. Moreover, the equilibrium of the serial contest is very robust.[6]

With respect to difference-form contests, the serial contest offers the advantages of weakening the absolute criterion in the assignment process and of being homogenous. It can also be easily defined for more than two contestants and we offer a preliminary equilibrium analysis of this case.[7] With respect to the robustness of the equilibrium

[6] The Tullock rent-seeking game has the drawback that it is only well understood when the CSF is either very insensitive to effort ($0 \leq \alpha \leq n/(n-1)$) or it is extremely sensitive ($\alpha \to \infty$). Baye *et al.* (1994) study the intermediate cases when two contestants have a common valuation for the prize. They characterize symmetric equilibria when the set of bids is discrete and analyze total expected rent dissipation. Concerning robustness, for the cases in which the Tullock rent-seeking game offers equilibrium predictions, these predictions differ from the all-pay auction qualitatively (see Nitzan (1994) and the discussion in Che and Gale (2000) or Fang (2002)).

[7] While Hirshleifer's (1989) difference-form contest is defined for any number of contestants, Che and Gale's CSF is not and it is not clear what the proper extension is.

predictions, Che and Gale (2000) show the convergence of equilibrium to the one of the polar all-pay auction, while in the serial contest equilibria are essentially the same for a wide range of non-determinacy.

The paper is organized as follows. We introduce the serial contest for any number of contestants in the next section. Section 3 analysis equilibrium in the case of two contestants. The robustness to an increase in the number of contestants is investigated in Section 4; and further links to the all-pay auction are derived in Section 5. The last section offers a concluding discussion.

2. The Serial Contest Model

Consider a contest where $n > 1$ agents (contestants) compete for a prize. Each contestant has a valuation for the object, denoted by V_i, and submits a bid b_i. The set of contestants or bidders is denoted by $\mathcal{B} = \{B_1, \ldots, B_i, \ldots, B_n\}$. Bidders are risk-neutral, and they bid simultaneously. The valuations are common knowledge and without loss of generality ordered such that $V_1 \geq V_2 \geq \ldots \geq V_n > 0$.

The winner is determined through a contest success function. This function associates, to each vector of bids b, a lottery specifying for each agent a probability of getting the object.

Definition 2.1. *[CSF] A contest success function is a mapping*

$$\Psi : \mathbb{R}^n \to \Delta^n$$

such that for each $b \in \mathbb{R}^n$, $\Psi(b)$ is in the $n-1$ dimensional simplex, i.e. $\Psi(b)$ is such that, for each i, $\Psi_i(b) \geq 0$, and $\sum_{i=1}^n \Psi_i(b) = 1$.

The CSFs (1.1), (1.2) and (1.4) mentioned in the Introduction are examples for Definition 2.1. Given the contest success function Ψ and linear costs of effort, agents' expected utility from participating in the contest, when the vector of bids is b, is

$$E\Pi_i(b) = \Psi_i(b) V_i - b_i. \tag{2.1}$$

We define now formally the class of serial contest functions for any number of contestants. In order to do this it is necessary to distinguish between non-degenerated ($\{b \geq 0, b \neq 0\}$) and degenerated bid vectors (all contestants bid zero). In the latter case we follow Baye *et al.* (1994) and establish a fair lottery for the prize.[8] Moreover, without loss of generality we suppose that the vector of bids is ordered such that

[8] This specification is also consistent in the sense that it follows from the requirements of an anonymous assignment and that probabilities add up to one. An alternative assumption is that the prize remains with the contest administrator but then the sum of probabilities is smaller than one.

$b_1 \geq b_2 \geq ... \geq b_n$.[9] Given this order, we can rewrite equation (1.4) as follows

$$\begin{aligned}\Psi_2(b) &= \frac{1}{2}\left(\frac{b_2}{b_1}\right)^{\alpha}\\ \Psi_1(b) &= \left[1-\left(\frac{b_2}{b_1}\right)^{\alpha}\right]+\frac{1}{2}\left(\frac{b_2}{b_1}\right)^{\alpha}.\end{aligned}$$

We are now in a position to extend the two-contestant CSF to any number of agents.

Definition 2.2. *[**Serial CSF**] If b is a non-degenerated and ordered vector of bids such that $b_1 \geq b_2 \geq ... \geq b_n \geq 0$. Then the serial contest success function with economics of scale parameter $\alpha \geq 0$ assigns for all $B_i \in \mathcal{B}$*

$$\Psi_i(b) = \sum_{j=i}^{n} \frac{b_j^{\alpha} - b_{j+1}^{\alpha}}{j \cdot b_1^{\alpha}},$$

with $b_{n+1} = 0$. If b is degenerated, then the serial contest success function establishes a fair lottery among the contestants.

It is common in the contest literature to interpret the scalar α as measuring economics of scale because it indicates the marginal return from lobbying efforts.

Notice that the class of serial contest success functions can also be defined recursively as follows

$$\Psi_n(b) = \frac{b_n^{\alpha}}{n \cdot b_1^{\alpha}} \quad \text{and} \quad \Psi_i(b) = \Psi_{i+1}(b) + \frac{b_i^{\alpha} - b_{i+1}^{\alpha}}{i \cdot b_1^{\alpha}} \text{ for all } i \in \{1, 2, \ldots, n-1\}.$$

3. The Two-Contestants Case

3.1. Constant Returns to Scale: $\alpha = 1$

In this section we analyze pure strategy Nash equilibria of the serial contest in its simplest functional form. We find that any equilibrium has all-pay auction properties, because contestants bid the expectation of the equilibrium mixed strategy of the all-pay auction. We establish a unique equilibrium.

Theorem 3.1. *Suppose $n = 2$, $V_1 \geq V_2$ and $\alpha = 1$. There exists a unique pure strategy Nash equilibrium to the serial contest. In this equilibrium contestants' bids*

[9] If necessary relabel the set of bidders.

and expected payoffs are as follows

$$b_1^* = \frac{V_2}{2} \qquad\qquad b_2^* = \frac{(V_2)^2}{2V_1}$$

$$E\Pi_1\,(b^*) = V_1 - V_2 \qquad E\Pi_2\,(b^*) = 0,$$

while total effort is

$$TE(b^*) = \frac{V_2(V_1+V_2)}{2V_1}.$$

Proof. See Appendix A.1. ■

The equilibrium has a particularly simple structure because the payoff function of B_2 is linear. For low b_1 it is increasing while for high b_1 it is decreasing. The equilibrium bid $b_1^* = V_2/2$ makes B_2 indifferent between any $b_2 \in [0, b_1^*]$. Given b_1^*, B_2's indifference means that he can as well bid b_2^*, which makes b_1^* optimal for B_1. Note that this indifference of the weaker contestant is also a feature of the equilibrium of the all-pay auction. In the all-pay auction the weaker bidder places the atom $(1 - V_2/V_1)$ at zero and randomizes continuously up to his valuation when he enters the contest. We show in Subsection 3.3 that decreasing α by an arbitrary small amount makes the indifference between abstaining and contesting disappear. But before doing so we close the gap between constant returns to scale and the polar case of the all-pay auction.

3.2. Increasing Returns to Scale: $\alpha > 1$

It is well known that in the all-pay auction there is no equilibrium in pure strategies. Moreover, non-deterministic contests (like the Tullock rent-seeking game) exhibit this feature when the CSF approaches the all-pay auction and becomes sensitive enough to effort. This is also true in the serial contest.

To see why there is no pure strategy equilibrium under increasing returns assume that there are only two bidders. At least one of them, say agent B_2, must choose b_2^* such that it maximizes $b_2^\alpha/(2b_1^\alpha)V_2 - b_2$. For $\alpha > 1$ this objective function is strictly convex and strictly negative for small bids. However, if b_1^* is low enough then the objective function becomes positive again and thus, $b_2^* = b_1^*$. But then, given the strong concavity of the CSF, it pays for bidder B_1 to increase his bid.

Proposition 3.2. *Suppose $\alpha > 1$. There exists no pure strategy Nash equilibrium to the serial contest.*

Proof. See Appendix A.2. ■

We turn now to mixed strategy equilibria. Such a strategy for player B_i is denoted by μ_i and the associated strategy profile is indicated by μ. The next result says that any serial contest in which the CSF is sufficiently sensitive to effort has an equilibrium with all-pay auction properties, since contestants bid in expected terms as in the deterministic case.

Theorem 3.3. *Suppose $n = 2$, $V_1 \geq V_2$ and $\alpha \geq 1$. There exists a Nash equilibrium in mixed-strategies to the serial contest. In this equilibrium contestants' expected bids and expected payoffs are as follows*

$$E(\mu_1^*) = \tfrac{V_2}{2} \qquad E(\mu_2^*) = \tfrac{(V_2)^2}{2V_1}$$

$$E\Pi_1(\mu^*) = V_1 - V_2 \qquad E\Pi_2(\mu^*) = 0,$$

while the expected total effort is

$$ETE(\mu^*) = \frac{V_2(V_1 + V_2)}{2V_1}.$$

Proof. See Appendix A.3. ■

3.3. Decreasing Returns to Scale: $\alpha < 1$

As for the constant returns to scale case, we establish a unique equilibrium. To underline the 'continuous' variation of equilibria for variations of α, the statement includes the one of Theorem 3.1.

Theorem 3.4. *Suppose $n = 2$, $V_1 \geq V_2$ and $\alpha \leq 1$. There exists a unique pure strategy Nash equilibrium to the serial contest. In this equilibrium contestants' bids and expected payoffs are as follows*

$$b_1^* = \tfrac{\alpha V_1}{2}\left(\tfrac{V_2}{V_1}\right)^{\alpha} \qquad b_2^* = \tfrac{\alpha V_2}{2}\left(\tfrac{V_2}{V_1}\right)^{\alpha}$$

$$E\Pi_1(b^*) = V_1\left[1 - \tfrac{\alpha+1}{2}\left(\tfrac{V_2}{V_1}\right)^{\alpha}\right] \qquad E\Pi_2(b^*) = V_2\tfrac{1-\alpha}{2}\left(\tfrac{V_2}{V_1}\right)^{\alpha},$$

while total effort is

$$TE(b^*) = \frac{\alpha}{2}\left(\frac{V_2}{V_1}\right)^{\alpha}(V_1 + V_2).$$

Proof. See Appendix A.4. ■

For $\alpha < 1$ both payoff functions are strictly concave and the equilibrium is strict. When the CSF becomes insensitive to effort and approaches the fair lottery, equilibrium bids go to zero.[10]

[10] It is interesting to observe that in this equilibrium total effort need not be increasing in α. Consider the following example. Let $V_1 = 9$ and $V_2 = 1$. We have that with $\alpha = 1/2$, $TE(b^*, \alpha = 1/2) = 5/6$. With $\alpha = 1$, $TE(b^*, \alpha = 1) = 5/9$.

4. Many Contestants

In this section we investigate the robustness of the equilibrium predictions of the two-bidder serial contest w.r.t an increase in the number of contestants. We find further similarities to the all-pay auction equilibrium when the CSF is not too insensitive to effort.

An important property of the all-pay auction refers to participation in the contest. A well known result says the following. Suppose valuations are decreasingly ordered, that is, $V_1 \geq V_2 \geq \cdots \geq V_n$. There exists a mixed strategy equilibrium in which the two contestants with the highest valuations bid in (expectation) as in Theorems 3.1 and 3.3. All other contestants abstain from the contest with probability one. Moreover, this equilibrium is unique if $V_2 > V_3$.[11] Thus, when embedding an all-pay auction into a larger game, one might argue that it is reasonable to deal only with two contestants. In this section we show that the serial contest has similar properties when the CSF is not too different from the all-pay auction. More precisely, this is true under non-decreasing returns to scale but not under decreasing returns to scale.

4.1. Constant Returns to Scale: $\alpha = 1$

The next result says that under constant returns to scale there is an equilibrium in pure strategies in which (loosely speaking) the prescriptions of Theorems 3.1 apply to contestants B_1 and B_2, while all other contestants have a strict incentive to abstain from the contest with probability one.

Theorem 4.1. *Suppose $V_1 \geq V_2 \geq \cdots \geq V_n$ and $\alpha = 1$. There exists a pure strategy Nash equilibrium to the serial contest. In this equilibrium contestants' bids and expected payoffs are as follows*

$$b_1^* = \tfrac{V_2}{2} \qquad b_2^* = \tfrac{V_2^2}{2V_1} \qquad b_j^* = 0 \text{ for } j > 2$$

$$E\Pi_1(b^*) = V_1 - V_2 \qquad E\Pi_2(b^*) = 0 \qquad E\Pi_j(b^*) = 0 \text{ for } j > 2,$$

while total effort is

$$TE(b^*) = \frac{V_2(V_1 + V_2)}{2V_1}.$$

[11] When there is a multiplicity of equilibria, in no equilibrium there is a contestant whose expected payoff exceeds the one specified in the statement. Moreover, the only case in which there is no revenue equivalence among equilibria is when more than one contestant have the second highest valuation which is strictly lower than the highest one. See Baye *et al.* (1996) for more details.

Proof. See Appendix A.5. ■

As the all-pay auction, the serial contest has multiple equilibria. For instance, if there are three contestants with the same valuation it is not clear who abstains. Moreover, even in situations in which valuations differ this multiplicity might persist. Consider the following example.

Example 4.2. *Let $(V_1, V_2, V_3) = (10, 9, 8)$ and $\alpha = 1$. The following strategy profile is an equilibrium. Contestant B_1 is not active and bids $\tilde{b}_1 = 0$. Bidder B_2 bids $\tilde{b}_2 = 4$ and B_3 exerts effort $\tilde{b}_3 = 32/9$. Given the abstention of B_1, the others act optimally (by Theorem 3.1). Given $\tilde{b}_2$ and $\tilde{b}_3$, $E\Pi_1(b_1, \tilde{b}_{-1}) < 0$ for all $b_1 > 0$.*

The strategy profile used in this example constitutes an equilibrium because of a 'coordination failure'. The 'wrong' set of contestants is active. Given the other bids and that valuations are very close it does not pay for the bidder with the highest valuation to submit a positive bid. This equilibrium disappears if V_1 increases. Despite the multiplicity of equilibria we can establish that an important property of the all-pay auction holds for any equilibrium of the constant returns to scale serial contest.

Theorem 4.3. *Suppose $V_1 \geq V_2 \geq \cdots \geq V_n$ and $\alpha = 1$. In any pure strategy Nash equilibrium of the serial contest there are exactly two active bidders. If a strategy profile constitutes an equilibrium in which the bidder pair B_i and B_k with $i < k$ is active, then the prescriptions of Theorem 3.1 apply (substituting the subscript 1 for i and 2 for k).*

Proof. See Appendix A.6. ■

The multiplicity of equilibria in the serial contest is a natural extension of the multiplicity of equilibria in the all-pay auction (see Theorem 1 and 2 in Baye *et al.* (1996)). In the latter a multiplicity arises when contestants have *exactly* the same valuation, while in the former a multiplicity exists when valuations are *exactly* or *almost* the same. In both contests different coordinations result in different sets of active contestants.

4.2. Increasing Returns to Scale: $\alpha > 1$

The next result says that we can extend the mixed strategy equilibrium in Theorem 3.3 in the same way as we did with the pure strategy equilibrium of Theorem 3.1.

Theorem 4.4. *Suppose $V_1 \geq V_2 \geq \cdots \geq V_n$ and $\alpha \geq 1$. There exists a Nash equilibrium in mixed-strategies to the n-player serial contest. In this equilibrium contestants' expected bids and expected payoffs are as follows*

$$E(\mu_1^*) = \tfrac{V_2}{2} \qquad E(\mu_2^*) = \tfrac{(V_2)^2}{2V_1} \qquad b_j^* = 0 \text{ for } j > 2$$

$$E\Pi_1(\mu^*) = V_1 - V_2 \qquad E\Pi_2(\mu^*) = 0, \qquad E\Pi_j(\mu^*) = 0 \text{ for } j > 2,$$

while the expected total effort is

$$ETE(\mu^*) = \frac{V_2(V_1 + V_2)}{2V_1}.$$

Proof. See Appendix A.7. ■

Again, there are multiple equilibria. To see this note that for more than three contestants Theorem 4.4 establishes an asymmetric equilibrium even if the game is symmetric. Following Dasgupta and Maskin (1986), our Lemma A.1 (in the Appendix) guarantees the existence of a symmetric equilibrium even when there are more than three contestants.

Because of Theorems 4.1 and 4.4 one might argue that it is reasonable to restrict the analysis of the serial contest to the case of only two bidders. We show next that this is no longer so when the CSF is very insensitive to effort.

4.3. Decreasing Returns to Scale: $\alpha < 1$

As in the constant returns to scale case, one might try to generalize the equilibrium of the two-bidder contest to any number of contestants. It turns out that this is a difficult task and we will leave an exhaustive analysis for future research. However, it is straightforward to see that with decreasing returns to scale low valuation contestants have strong incentives not to abstain from the contest.

Taking into account that valuations are decreasingly ordered, it is natural that this order is reflected in the bids of a pure strategy equilibrium. The contestants' first order conditions imply then that the following strategy profile is a candidate equilibrium

$$\begin{aligned} b_1^* &= \left[\alpha V_1 \left(\sum_{j=2}^{n} \frac{1}{j(j-1)} \left(\frac{\alpha V_j}{j}\right)^{\frac{\alpha}{1-\alpha}}\right)\right]^{1-\alpha} \quad \text{and} \\ b_i^* &= \left(\frac{\alpha V_i}{i}\right)^{\frac{1}{1-\alpha}} \left[\alpha V_1 \left(\sum_{j=2}^{n} \frac{1}{j(j-1)} \left(\frac{\alpha V_j}{j}\right)^{\frac{\alpha}{1-\alpha}}\right)\right]^{-\alpha}, \forall B_i \neq B_1. \end{aligned}$$

Note that for $n = 2$, this strategy profile becomes identical to the one in Theorem 3.4.[12] Notice also that $b_i^* > 0$ for all $B_i \notin \{B_1, B_2\}$. It turns out that in many situations

[12] It can also be shown that the candidate equilibrium strategy profile converges to the one for constant returns to scale. That is, as $\alpha \to 1^-$, $b_1^* \to \frac{V_2}{2}, b_2^* \to \frac{(V_2)^2}{2V_1}$ and $b_i^* \to 0$ for all agents B_i with $i > 2$.

the candidate equilibrium constitutes indeed a pure strategy equilibrium to the many-bidder serial contest with decreasing returns to scale. It is thus straightforward to find examples in which more than two contestants have an incentive to participate.[13]

To summarize, there are situations in which the predictions of the many-bidder serial contest might differ from the two-contestant case. However, it is worth pointing out that this might happen when the CSF is very insensitive to effort. These situations are the less interesting cases because the applications of contest models mentioned in the Introduction are not instances in which contestants have very limited influence in determining the winner of the contest.

5. Properties of Equilibrium: Further Links to the All-Pay Auction

In this section we show that – apart from the links we have already established between the serial contest and the all-pay auction – there are important further properties which both share.

5.1. The Extent of Rent Dissipation

Apart from the issue of existence of Nash equilibria in pure and mixed strategies, the primary concern of the rent-seeking literature has been the question how different contests affect rent dissipation. As usual, assume in this subsection that all agents have the same valuation V for the political prize. The rent dissipation rate D is measured by the ratio between total rent-seeking outlays in equilibrium and the value of the contested rent.

Corollary 5.1. *Assume $n = 2$ and $V_1 = V_2 = V$. In (a symmetric) equilibrium of the serial contest the extent of rent dissipation is $D = min\{\alpha, 1\}$.*

Proof. See Appendix A.8. ■

The corollary shows that the serial contest shares with the all-pay auction the following feature: with symmetric valuations (and $\alpha \geq 1$) the rent is fully dissipated, even when the number of rent-seekers is small. Moreover, this conclusion remains approximately true for further values of the economies of scale parameter α. This contrasts with a well known result establishing $D = 1/2$ in the Tullock rent-seeking game with economics of scale parameter $\alpha = 1$ (see e.g. Konrad (2006)).

[13] It is, however, in general not true that the candidate strategy profile is always an equilibrium. When valuations are close, it is possible to construct counter-examples. In these examples, because of the strong concavity of the CSF, lower bidders have an incentive to outbid the highest bidder.

5.2. The Exclusion Principle

In their analysis of the all-pay auction Baye *et al.* (1993) have identified an interesting incentive for a contest administrator that is very related to the extent of rent dissipation. They consider an administrator who is interested in maximizing the expected total amount of bids. The *exclusion principle* is defined as the precommitment to preclude contestants most valuing the prize from participating in the contest. Baye *et al.* (1993) show that depending on the vector of valuations the administrator may have an incentive to organize such an inefficient contest. Given that our Theorems 4.1 and 4.4 establish that the expected equilibrium revenues follow equation (10) in Baye *et al.* (1993), we can directly apply their Proposition 2 and obtain the following result.

Corollary 5.2. *Let $V_1 \geq V_2 \geq \cdots \geq V_n$. For any $\alpha \geq 1$, there exist valuations such that the exclusion principle applies and the politician benefits from excluding the contestants valuing the prize most from participating in the serial contest.*

Notice that the exclusion principle does not apply to the Tullock rent-seeking game with economics of scale parameter $\alpha = 1$ (Fang (2002)).

5.3. The Preemption Effect

Another related property of the all-pay auction is the preemption effect (see Che and Gale (2000)). This effect occurs if an increase in the asymmetry of valuations of contestants causes the low valuation bidder to be more pessimistic about his prospects of winning and to become less aggressive. This allows the high valuation bidder to bid less aggressive, too. We define the *preemption effect* as a decrease in expected total effort due to a decrease in V_2.[14] Direct computation of $\partial ETE/\partial V_2$ from the expressions in Theorems 3.3 and 3.4 shows that this derivative is positive and we have the following result.

[14] We define the preemption effect in response to a variation in V_2 in order to fit it with the preceding intuition from Che and Gale (2000). These authors define the effect as a decrease in expected total effort due to an increase in V_1 (p. 37). For the case of the all-pay auction under both definitions there is preemption. However, B_1 does not lower his bid on average in response to an increase in V_1. Instead, his expected bid remains unchanged. To the contrary, under our definition both bidders bid less aggressive. For the serial contest the choice of definition matters. Defining the effect in response to V_1 yields that for $\alpha \geq 1$, there is always a preemption effect. For $\alpha \in (\frac{1}{2}, 1)$, there is a preemption effect if and only if $\alpha > \frac{V_1}{V_1+V_2}$ and for $\alpha \in [0, \frac{1}{2}]$, there is never a preemption effect. Broadly speaking, the lower α, the more moderate the asymmetry must be for the preemption effect to occur. The reason for the different results under both definitions comes from the fact that in response to a change in V_1 contestant B_1 bids more aggressive while B_2 reduces his effort. For the preemption effect to apply, the increase must be less than the reduction. In the second price all-pay auction the occurrence of a preemption effect depends also on the way the asymmetry is increased (see Riley (1999)).

Corollary 5.3. *Assume $n = 2$ and $V_1 \geq V_2$. For any $\alpha > 0$, there is always a preemption effect in the serial contest.*

5.4. The Effect of a Cap on Political Lobbying

Che and Gale (1998) use the all-pay auction model to study campaign spending in a lobbying game. They analyze the effect of contribution limits – modelled by a cap m on bids – on aggregate expenditures. They show that a cap on individual bids may have the perverse effect of increasing expected total effort. The intuition for this effect is that a cap can attenuate bidder 1's ability to preempt bidder 2. As a result bidding competition may be increased and aggregate expenditures may be raised. A similar result is true in the serial contest.

Broadly speaking, given that the cap is low enough, in the equilibrium of the serial contest both contestants submit the highest possible bid. Thus, the restricted ability to preempt results in a game that admits a pure strategy Nash equilibrium – even when without cap there is none (see Proposition 3.2). These strategies are not an equilibrium without a cap, because for B_1 it pays to outbid the competitor. But this deviation from the candidate equilibrium is prevented by the cap, making the strategy profile an equilibrium. A cap increases expected total effort if valuations are asymmetric enough and, as a consequence, in the equilibrium without a cap the total effort is relatively low. We obtain the following result.

Corollary 5.4. *Assume $n = 2$ and $V_1 > V_2$. For any $\alpha > 0$, a cap has always the potential to increase expected total effort in the serial contest. Formally, this occurs if*

$$\frac{\beta}{4}\left(\frac{V_2}{V_1}\right)^{\beta}(V_1 + V_2) < m < \frac{\beta V_2}{2}, \tag{5.1}$$

where $\beta = \alpha$ if $\alpha \leq 1$ and $\beta = 1$ otherwise.

Proof. See Appendix A.9. ∎

This contrasts with Fang's result that a cap does not increase expected total effort in the Tullock rent-seeking game with economics of scale parameter $\alpha = 1$ (Fang (2002)).

6. Concluding Remarks

We have analyzed a family of contest games in which the sensitivity of the contest success function to the contestants' efforts is parameterized by a economics of scale parameter α. It contains the polar cases of the (fair) lottery and the (deterministic) all-pay auction in which $\alpha = 0$ and $\alpha \to \infty$, respectively. Our model has advantages over previous work on contests.

First, although win probabilities depend on effort differences, we weaken this absolute criterion. In the serial contest win probabilities depend on percentage mark-ups of effort. Moreover, the serial contest is homogenous of degree zero in effort.

Second, in applications, two-player contests are often plugged into larger models in order to analyze the effect of conflict situations on other variables of interest. In such a situation the game is solved by backwards induction and the equilibrium payoffs of the contest subgame are plugged into the previous stage. The serial contest offers a very tractable model for this purpose. We have shown that given contestants' valuations V_1 and V_2, with $V_1 \geq V_2$, equilibrium payoffs are given by

$$E\Pi_1 = V_1 \left[1 - \tfrac{\beta+1}{2} \left(\tfrac{V_2}{V_1} \right)^{\beta} \right] \quad \text{and} \quad E\Pi_2 = V_2 \tfrac{1-\beta}{2} \left(\tfrac{V_2}{V_1} \right)^{\beta},$$

where $\beta = \alpha$ if $\alpha \leq 1$ and $\beta = 1$ otherwise.[15] Thus, the model builder does not need to deal with the specific algebraic form of the equilibrium strategies. These expressions apply to the polar case of the all-pay auction and finite values for α can be interpreted as departures from this polar case through a contest success function less and less sensitive to effort. The model builder can thus easily check for robustness, that is, up to which value for α the conclusions of the polar all-pay auction remain true. Further advantages are that this equilibrium is unique in many situations and that often further contestants have a strict incentive not to participate in the contest.

Third, in their analysis of the (two-player) difference-form contest, Che and Gale (2000) provide important robustness results for the all-pay auction in the sense that the equilibrium converges to that of the all-pay auction. In contrast, this paper provides robustness results for the all-pay auction because equilibria (with any number of contestants) are essentially the same over a large parameter region.

While our model is a step toward a general theory of contests, more work is needed. On one hand, there are open questions within the class of serial contests. First, for increasing returns to scale, we do not offer an explicit derivation of the equilibrium mixed-strategies and it is important to determine the properties of other equilibria when they exist. Second, our analysis of the many player contest for decreasing returns to scale is incomplete and it is important to know what the equilibrium is. On the other hand, given that the serial contest shares important properties of equilibrium with other contests, it is important to know whether these properties extend to more general classes of contests.

[15] To the best of our knowledge, for the popular Tullock rent-seeking game it is not known what the equilibrium is when there is no common value and the economics of scale parameter α is larger than $n/(n-1)$. In our companion paper Alcalde and Dahm (2007) we tackle this issue generalizing the technics used in the current paper.

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A. Appendix

A.1. Proof of Theorem 3.1

We show first that the profile b^* is an equilibrium. Suppose B_2 bids b_2^*. We have that $E\Pi_1(b_1, b_2^*) = b_1\Big(\left(\frac{V_1}{V_2}\right)^2 - 1\Big)$ if $0 \leq b_1 \leq b_2^*$ and $E\Pi_1(b_1, b_2^*) = V_1 - \frac{(V_2)^2}{4b_1} - b_1$ otherwise. Note that b_1^* maximizes this function. Suppose B_1 bids b_1^*. We obtain $E\Pi_2(b_1^*, b_2) = 0$ if $0 \leq b_2 \leq b_1^*$ and $E\Pi_2(b_1^*, b_2) = V_2 - \frac{(V_2)^2}{4b_2} - b_2 \leq 0$ otherwise. Hence, B_2 can not do better than bidding b_2^*.
We show now uniqueness. Suppose that there is another equilibrium $b^{**} \neq b^*$. Because of the preceding we have $b_1^{**} \neq b_1^*$ and $b_2^{**} \neq b_2^*$. First, suppose $b_2^{**} < b_1^{**}$. We have that $E\Pi_2(b^{**}) = b_2^{**}\left(\frac{V_2}{2b_1^{**}} - 1\right)$. If $b_1^{**} < b_1^*$, then $E\Pi_2(b^{**})$ is increasing in b_2^{**} implying $b_2^{**} = b_1^{**}$. If $b_1^{**} > b_1^*$, then $E\Pi_2(b^{**})$ is decreasing in b_2^{**} implying $b_2^{**} = 0$. But then B_1 can improve by bidding b_1^*. Second, assume $b_2^{**} \geq b_1^{**}$. We have that $E\Pi_1(b^{**}) = b_1^{**}\left(\frac{V_1}{2b_2^{**}} - 1\right)$ and $E\Pi_2(b^{**}) = V_2 - \frac{b_1^{**}V_2}{2b_2^{**}} - b_2^{**}$. The maximizer of $E\Pi_2(b^{**})$ is $b_2^{**} = \sqrt{\frac{b_1^{**}V_2}{2}}$. If $b_1^{**} < b_1^*$, this implies that $E\Pi_1(b^{**})$ is increasing in b_1^{**} yielding the contradiction $b_1^{**} = b_2^{**} = b_1^*$. If $b_1^{**} > b_1^*$, then at the optimum we have $E\Pi_2(b^{**}) = V_2 - 2\sqrt{\frac{b_1^{**}V_2}{2}} < 0$.
Straightforward computations yield $E\Pi\left(b^*\right)$ and $TE(b^*)$. *Q.E.D.*

A.2. Proof of Proposition 3.2

Suppose b^* is a pure strategy equilibrium. Analogously to the reasoning in step 1 of the proof of Theorem 4.3 it can be established that the cardinality of the set of active bidders $|\mathcal{B}^{\mathcal{A}}| \geq 2$. W.l.o.g. number these contestants such that $b_1^* \geq b_2^* \geq \cdots \geq b_k^* > 0$. Consider any $B_i \in \mathcal{B}^{\mathcal{A}}$ with $i \geq 2$. Equilibrium requires $E\Pi_i(b^*) = \left[\Psi_{i+1} + \frac{(b_i^*)^\alpha - (b_{i+1}^*)^\alpha}{i(b_1^*)^\alpha}\right]V_i - b_i^* = \left[\Psi_{i+1} - \frac{(b_{i+1}^*)^\alpha}{i(b_1^*)^\alpha}\right]V_i + b_i^*\left[\frac{(b_i^*)^{\alpha-1}}{i(b_1^*)^\alpha}V_i - 1\right] \geq 0$ (setting $\Psi_{n+1} = 0$). Given that the first term is negative, it is needed that the second is positive. Given $\alpha > 1$, the latter inequality implies that $\frac{\partial E\Pi_i(b_i, b_{-i}^*)}{\partial b_i} = \alpha\frac{(b_i^*)^{\alpha-1}}{i(b_1^*)^\alpha}V_i - 1 > \frac{(b_i^*)^{\alpha-1}}{i(b_1^*)^\alpha}V_i - 1 \geq 0$ holds, $\forall b_i \in [b_i^*, b_{i-1}^*]$. Thus, $b_1^* = b_2^* = \cdots = b_k^* > 0$ must hold. $E\Pi_i(b^*) \geq 0$ implies $b_1^* \leq \frac{V_i}{k}$ for all $i \in \{1, \ldots, k\}$. Consider the active bidder with the highest valuation, say, B_j. If $b_j \geq b_1^*$, then $E\Pi_j(b_j, b_{-j}^*) = \left[1 - \frac{(b_1^*)^\alpha}{k(b_j^*)^\alpha}\right]V_j - b_j$ and $\left.\frac{\partial E\Pi_j(b_j, b_{-j}^*)}{\partial b_j}\right|_{b_j = b_1^*} > 0 \Leftrightarrow b_1^* < \frac{\alpha V_j}{k}$.

Given $\alpha > 1$ and that V_j is the highest valuation, the latter holds and B_j has a profitable deviation. *Q.E.D.*

A.3. Proof of Theorem 3.3

The proof we offer is constructive and uses three lemmata. In a first step, we use a finite approximation to establish the existence of a symmetric mixed strategy equilibrium for any symmetric serial contest and to derive some properties of it. We go then a step further and build on this equilibrium to obtain one in asymmetric contests.[16]

We start by normalizing valuations such that $(\hat{V}_1, \hat{V}_2) = (1, V_2/V_1)$. Since the serial contest is homogenous of degree zero in effort, this is w.l.o.g., because it does not change the ranking of payoffs from pure strategies. If a pure strategy b_1/V_1 yields

$$E\Pi_1(b_1/V_1, \mu_2^*) = Prob\{B_1 \; wins \mid \mu_2^*\} - \frac{b_1}{V_1}$$

in the normalized game, then b_1 obtains $E\Pi_1(b_1, \mu_2^*) = E\Pi_1(b_1/V_1, \mu_2^*)V_1$ in the original game because win probabilities are the same.

Consider first the symmetric serial contest in which there is a common valuation $\hat{V} = 1$. We consider in addition to the original game a discrete version of the serial contest. The serial contest is *finite with grid G* if the strategy space is discrete such that given some integer G only bids on the finite grid $\{0, 1/G, 2/G, \ldots, (G-1)/G, 1\}$ are feasible. We indicate an arbitrary element of the grid by x/G where $x \in \{0, 1, \ldots, G\}$.[17] Let $\tilde{\mu}^G$ denote an equilibrium to the game with finite grid G. The first lemma is a direct consequence of Dasgupta and Maskin (1986). It establishes existence of equilibrium and relates the continuous to the finite game. For later reference it refers to any number of contestants.[18]

Lemma A.1. *The symmetric n-bidder serial contest has a symmetric mixed strategy Nash equilibrium, both when the strategy space is finite and when it is continuous. Moreover, the profile $\mu^* = lim_{G\to\infty}\, \tilde{\mu}^G$ exists and constitutes a mixed strategy Nash equilibrium to the continuous serial contest.*

Proof. The existence of a symmetric equilibrium for the serial contest with finite grid G follows from Lemma 6 in Dasgupta and Maskin (1986). We will show that the conditions of their Theorem 6 are also satisfied. This theorem guarantees the existence

[16] Our analysis of the symmetric serial contest follows very closely Baye *et al.* (1994) who investigate the symmetric two-bidder Tullock rent-seeking contest.

[17] Note that this is realistic because it implies the existence of a smallest monetary unit $1/G$, like in experimental settings. For simplicity we say that bidder B_i bids x, although we mean x/G.

[18] From Dasgupta and Maskin's results it also follows that the equilibrium strategy has no atom at zero effort, since this is a point of discontinuity of the CSF.

of a symmetric mixed strategy equilibrium when the strategy space is continuous. In addition, the proof of Dasgupta and Maskin's Theorem 6 shows that the limiting equilibrium of a finite approximation to the strategy space as the grid size goes to zero is indeed an equilibrium to the continuous game. The application of their theorem requires five conditions to be fulfilled. First, the sum of payoffs must be upper semi-continuous. Since $\sum_{i=1}^{n} E\Pi_i(b) = 1 - \sum_{i=1}^{n} b_i$ is continuous, it is upper semi-continuous, too. Second, $E\Pi_i(b)$ must be bounded. This holds as $-1 \leq E\Pi_i(b) \leq 1$ for $b_i \in [0,1]$ and $i = 1, 2, \ldots, n$. Third, the discontinuities in the individual payoff functions must be of dimension less than n. This holds because for the serial contest the origin is the only point of discontinuity. Fourth, a so-called property α must hold. Because of the single discontinuity this inequality is straightforward to check. Fifth, the individual payoff functions $E\Pi_i(b_i, b_{-i})$ must be weakly lower semi-continuous in b_i. This is fulfilled as they are lower semi-continuous. Thus, Theorem 6 in Dasgupta and Maskin (1986) can be applied. ■

We return now to two-bidder serial contest with finite grid G.

Lemma A.2. *Suppose $\alpha \geq 1$. In any symmetric Nash equilibrium $\tilde{\mu}^G = (\tilde{\mu}_1^G, \tilde{\mu}_2^G)$ of a symmetric two-bidder serial contest with finite grid G it is true that*

$$\textit{(1) } 0 \leq E\Pi_i(\tilde{\mu}^G) \leq \frac{1}{G} \textit{ and } (2) E(\tilde{\mu}_i^G) = \frac{1}{2} - E\Pi_i(\tilde{\mu}^G).$$

Proof. First of all, let us introduce some additional notation. Given G, and contestant B_i's strategy $\tilde{\mu}_i^G$, $\tilde{\mu}_{ik}^G$ denotes the probability that contestant B_i assigns to bidding k/G. Note that in a symmetric equilibrium no mass will be placed at 1, that is, $\tilde{\mu}_{iG}^G = 0$ for $i = 1, 2$. To proof Lemma A.2, we will concentrate on agent B_1. A similar reasoning applies to agent B_2.

(1) (a) For the lower bound: The expected payoff from bidding x when the opponent follows the equilibrium strategy $\tilde{\mu}_2^G$ is

$$E\Pi_1(x, \tilde{\mu}_2^G) = \sum_{j=0}^{G-1} \tilde{\mu}_{2j}^G \Psi_1(x, j) - \frac{x}{G}. \tag{A.1}$$

Choosing $x = 0$ a contestant can secure himself $E\Pi_1(0, \tilde{\mu}_2^G) \geq 0$. Thus, $E\Pi_i(\tilde{\mu}^G) \geq 0$.

(b) For the upper bound: Given that $\tilde{\mu}^G$ is an equilibrium, B_1 must react optimally to B_2's strategy. Hence, the following holds for all x: (i) $E\Pi_1(x, \tilde{\mu}_2^G) \leq E\Pi_1(\tilde{\mu}^G)$, (ii) $E\Pi_1(x, \tilde{\mu}_2^G) = E\Pi_1(\tilde{\mu}^G)$ if $\tilde{\mu}_{1x}^G > 0$ and (iii) $\tilde{\mu}_{1x}^G = 0$ if $E\Pi_1(x, \tilde{\mu}_2^G) < E\Pi_1(\tilde{\mu}^G)$. Using (A.1), condition (i) can be rewritten as

$$\sum_{j=0}^{G-1} \tilde{\mu}_{2j}^G \Psi_1(x, j) \leq E\Pi_1(\tilde{\mu}^G) + \frac{x}{G}. \tag{A.2}$$

Let $x \geq 0$ be the lowest bid that is part of the symmetric mixed strategy equilibrium.[19] By (ii) condition (A.2) holds with equality

$$\frac{1}{2}\tilde{\mu}^G_{2x} + \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\tilde{\mu}^G_{2(x+1)} + \cdots + \frac{1}{2}\frac{x^\alpha}{(G-1)^\alpha}\tilde{\mu}^G_{2(G-1)} = E\Pi_1(\tilde{\mu}^G) + \frac{x}{G}. \tag{A.3}$$

For $x+1$ condition (A.2) becomes

$$\left(1 - \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\right)\tilde{\mu}^G_{2x} + \frac{1}{2}\tilde{\mu}^G_{2(x+1)} + \cdots + \frac{1}{2}\frac{(x+1)^\alpha}{(G-1)^\alpha}\tilde{\mu}^G_{2(G-1)} \leq E\Pi_1(\tilde{\mu}^G) + \frac{x+1}{G}. \tag{A.4}$$

Computing $\tilde{\mu}^G_{2x}$ from equation (A.3) and substitution in inequality (A.4) yields

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{x^\alpha}{(x+1)^\alpha}\left(1 - \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\right)\right)\tilde{\mu}^G_{2(x+1)} + \\
&+ \left(\frac{1}{2}\frac{(x+1)^\alpha}{(x+2)^\alpha} - \frac{x^\alpha}{(x+2)^\alpha}\left(1 - \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\right)\right)\tilde{\mu}^G_{2(x+2)} + \\
&+ \cdots + \left(\frac{1}{2}\frac{(x+1)^\alpha}{(G-1)^\alpha} - \frac{x^\alpha}{(G-1)^\alpha}\left(1 - \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\right)\right)\tilde{\mu}^G_{2(G-1)} \leq \\
&\leq E\Pi_1(\tilde{\mu}^G) + \frac{x+1}{G} - \left(2E\Pi_1(\tilde{\mu}^G) + \frac{2x}{G}\right)\left(1 - \frac{1}{2}\frac{x^\alpha}{(x+1)^\alpha}\right) = \\
&= \frac{1}{G}\left(1 - x\left(1 - \frac{x^\alpha}{(x+1)^\alpha}\right)\right) - E\Pi_1(\tilde{\mu}^G)\left(1 - \frac{x^\alpha}{(x+1)^\alpha}\right).
\end{aligned} \tag{A.5}$$

Note that every term on the left hand side of condition (A.5) is non-negative. To see this, define $\xi = \frac{x^\alpha}{(x+1)^\alpha} \in [0,1]$. Each term is non-negative if and only if $\xi^2 - 2\xi + 1 \geq 0$, which is true. Suppose, by way of contradiction, that $E\Pi_1(\tilde{\mu}^G) > \frac{1}{G}$. The right hand side of condition (A.5) is strictly smaller than

$$\frac{1}{G}\left(\frac{x^\alpha}{(x+1)^\alpha} - x\left(1 - \frac{x^\alpha}{(x+1)^\alpha}\right)\right) \leq 0 \Leftrightarrow x \leq x+1.$$

(2) We have that in a symmetric equilibrium

$$E\Pi_i(\tilde{\mu}^G) = Prob\{B_i\ wins\} - E(\tilde{\mu}^G_i).$$

Summing up for both agents gives

$$2E\Pi_i(\tilde{\mu}^G) = Prob\{B_1\ wins\} + Prob\{B_2\ wins\} - 2E(\tilde{\mu}^G_i)$$

and rearranging yields the statement. ■

[19] I.e., $\tilde{\mu}^G_{1x} > 0$, and $\tilde{\mu}^G_{1j} = 0$ for all $j < x$.

Lemma A.3. *Suppose $\alpha \geq 1$. If μ^* is a symmetric (possibly mixed) Nash equilibrium strategy profile of a symmetric two-bidder serial contest, then the following bidding strategies ν^* constitute a Nash equilibrium to the asymmetric two-bidder serial contest with $\hat{V}_1 > \hat{V}_2$:*

- *Contestant B_1 bids $\nu_1^* = \mu_1^*$ and*
- *contestant's B_2's strategy ν_2^* is such that he abstains from the contest with probability $(1 - \hat{V}_2/\hat{V}_1)$ and bids μ_2^* whenever he participates.*

Proof. To see that B_2 has no profitable deviation from ν_2^*, note that in the symmetric game B_2 obtains an expected payoff of $E\Pi_2(\mu^*) = 0$. Since $\nu_1^* = \mu_1^*$ and $\hat{V}_2$ is the same in both games, any pure strategy yields the same as in the symmetric game, B_2 obtains $E\Pi_2(\nu^*) = 0$ and is willing to abstain with probability $(1 - \hat{V}_2/\hat{V}_1)$. For B_1 note that in the symmetric game, given the mixed strategy μ_2^* by B_2, all pure strategies b_1 in the support of μ_1^* maximize

$$E\Pi_1(b_1, \mu^*) = \hat{V}_2 E[\Pr\{B_1 \text{ wins}|b_1, \Psi_1, \mu^*\}] - b_1, \tag{A.6}$$

where $E[\Pr\{B_1 \text{ wins}|b_1, \Psi_1, \mu^*\}]$ is B_1's expected win probability from the pure strategy b_1 when the CSF is Ψ_1 and B_2 mixes according to the equilibrium strategy μ^*. Note that, although we do not know whether μ^* is a continuous, discrete, or partially continuous and discrete distribution, the following must be true. When $\hat{V}_1 > \hat{V}_2$, since $\nu_2^* = \mu_2^*$ (conditional on entry), we look for pure strategies that maximize

$$E\Pi_1(b_1, \mu_2^*) = \left(1 - \frac{\hat{V}_2}{\hat{V}_1}\right)\hat{V}_1 + \hat{V}_2 E[\Pr\{B_1 \text{ wins}|b_1, \Psi_1, \mu^*\}] - b_1. \tag{A.7}$$

Given that any pure strategy in the support of μ_1^* maximizes (A.6) and that (A.6) and (A.7) only differ by an additive constant, any pure strategy in the support of μ_1^* maximizes (A.7), too. For later reference observe that a pure strategy maximizing (A.6) yields $E\Pi_2(\mu^*) = 0$, while with (A.7) $E\Pi_1(\nu^*) = \hat{V}_1 - \hat{V}_2$ is obtained. ■

We are now in a position to prove Theorem 3.3.
Proof of Theorem 3.3. The existence of a mixed strategy equilibrium follows from Lemma A.1. The expressions for expected bids, total effort and payoffs follow from Lemmata A.2 and A.3, taking into account that we normalized the game. *Q.E.D.*

Although the explicit derivation of the equilibrium mixed-strategies is beyond the scope of the present paper, we conclude this proof computing four examples of the symmetric two-bidder serial contest with a finite strategy space. We represent the cases of α equal to 1, 2, 10 and ∞ with a grid of $G = 13$ in Figure A.1. The computations

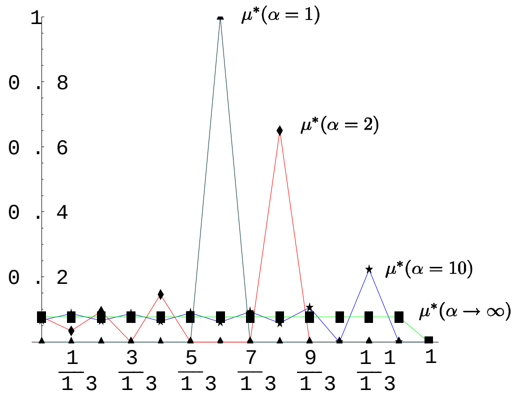

Figure A.1: $Q = 13$ and $\alpha \in \{1, 2, 10, \infty\}$

suggest that, as the returns to scale increase, the bulk of probability mass shifts to the right and some mass is attached to low bids. As α increases further, μ^* becomes more and more uniformly distributed, which is the optimal bidding strategy in the all-pay auction.[20]

A.4. Proof of Theorem 3.4

For $\alpha = 1$, the statement follows from Theorem 3.1. Suppose $\alpha < 1$. Given two bids b_H, b_L with $b_H \geq b_L$, define the maximization problems $[P_L]$ $max_{b_L}\ \frac{1}{2}\left(\frac{b_L}{b_H}\right)^{\alpha} V_L - b_L$ and $[P_H]$ $max_{b_H}\ \left(1 - \frac{1}{2}\left(\frac{b_L}{b_H}\right)^{\alpha}\right) V_H - b_H$, with unique maximizers $\tilde{b}_L(b_H) = \left(\frac{2}{\alpha V_L} b_H^{\alpha}\right)^{\frac{1}{\alpha-1}}$ and $\tilde{b}_H(b_L) = \left(\frac{\alpha V_H}{2} b_L^{\alpha}\right)^{\frac{1}{\alpha+1}}$, respectively.
We show first that the profile b^* is an equilibrium. Suppose B_2 bids b_2^*. We have that if $b_1 \geq b_2^*$, then the optimal choice must solve $[P_H]$ and we obtain $\tilde{b}_H(b_2^*) = b_1^*$. If $b_1 \leq b_2^*$, then b_1 must solve $[P_L]$ and $\tilde{b}_L(b_2^*) = \frac{\alpha}{2} V_1^{\frac{-1}{\alpha-1}} V_2^{\frac{\alpha}{\alpha-1}} \left(\frac{V_2}{V_1}\right)^{\frac{\alpha^2}{\alpha-1}}$. This means that B_1 has an incentive to raise his bid until equalling the one of B_2, because $\tilde{b}_L(b_2^*) \geq b_2^*$ if and only if $\left(\frac{V_2}{V_1}\right)^{\frac{\alpha+1}{\alpha-1}} \geq 1$ which is true. Suppose B_1 bids b_1^*. We have that if $b_2 \geq b_1^*$, then the

[20] Due to the finiteness, contestants obtain very low but strictly positive expected profits (smaller than 0.039). Moreover, the expected bid – even of the constant returns to scale case and the discrete all-pay auction – is strictly lower than 0.5 (but larger than 0.46). Baye et all (1994) have shown that in the two-player case the symmetric equilibrium of the discrete all-pay auction converges to the unique equilibrium of the continuous strategy space all-pay auction.

solution to $[P_H]$ and yields $\tilde{b}_H(b_1^*) = \frac{\alpha}{2} V_1^{\frac{\alpha}{\alpha+1}} V_2^{\frac{1}{\alpha+1}} \left(\frac{V_2}{V_1}\right)^{\frac{\alpha^2}{\alpha+1}}$. This implies that that B_2 has an incentive to lower his bid until equalling the one of B_1, because $\tilde{b}_H(b_1^*) \leq b_1^*$ if and only if $\left(\frac{V_2}{V_1}\right)^{1-\alpha} \leq 1$ which is true. If $b_2 \leq b_1^*$, then solving $[P_L]$ the optimal choice is $\tilde{b}_L(b_1^*) = b_2^*$.
Uniqueness follows from the fact that any other equilibrium $b^{**} \neq b^*$ must be a different solution to $[P_L]$ and $[P_H]$. However, $\tilde{b}_H = b_1^*$ and $\tilde{b}_L = b_2^*$ are the only solution satisfying $b_1^* \geq b_2^*$. Straightforward computations yield $E\Pi\left(b^*\right)$ and $TE(b^*)$. *Q.E.D.*

A.5. Proof of Theorem 4.1

Consider the pure strategy equilibrium of Theorem 3.1. This result implies that given $b_j^* = 0$ for $j > 2$, the efforts of B_1 and B_2 are optimal. Consider any B_j with $j > 2$. We have that $E\Pi_j(b_j, b_{-j}^*) = b_j\left(\frac{2V_j}{3V_2} - 1\right)$ if $0 \leq b_j \leq b_2^*$, $E\Pi_j(b_j, b_{-j}^*) = \left(\frac{b_j}{V_2} - \frac{V_2}{6V_1}\right)V_j - b_j$ if $b_2^* \leq b_j \leq b_1^*$ and $E\Pi_j(b_j, b_{-j}^*) = \left(1 - \frac{V_2}{4b_j}\left(1 + \frac{V_2}{3V_1}\right)\right)V_j - b_j$ otherwise. In all three cases $E\Pi_j(b_j, b_{-j}^*) < 0$. *Q.E.D.*

A.6. Proof of Theorem 4.3

We show first that in any equilibrium there are exactly two active bidders.
Suppose b^* is an equilibrium. Denote $b_H^* = max_{i=1,\ldots,n} b_i^*$ and $b_L^* = min_{j=1,\ldots,n} b_j^*$ s.t. $b_j^* > 0$. The sets $\mathcal{B}^A$, $\mathcal{B}^H$ and $\mathcal{B}^L$ denote the set of active bidders, the set of bidders with effort b_H^* and the contestants who bid b_L^*, respectively. The cardinality of these sets is denoted by $|\mathcal{B}^A|$, $|\mathcal{B}^H|$ and $|\mathcal{B}^L|$.
Step 1: $|\mathcal{B}^A| \geq 2$.
If $|\mathcal{B}^A| = 0$, then for all $\mathcal{B}_i$ holds $E\Pi_i(b^*) = \frac{V_i}{n}$, but $E\Pi_i(b_i = \frac{V_i}{2n}, b_{-i}^*) = \frac{2n-1}{2n}V_i > E\Pi_i(b^*)$. If $|\mathcal{B}^A| = 1$, then the active bidder can improve by bidding $\frac{b_H^*}{2}$.
Step 2: $\mathcal{B}^A = \mathcal{B}^H \cup \mathcal{B}^L$.
Suppose $\exists b_i^*$ s.t. $b_H^* > b_i^* > b_L^*$ and let b_i^* be the lowest such bid. We have that $E\Pi_i(b^*) = V_i\left(\frac{b_i^* - b_L^*}{(|\mathcal{B}^A| - |\mathcal{B}^L|)b_H^*} + \frac{b_L^*}{|\mathcal{B}^A|b_H^*}\right) - b_i^*$. Since this is a linear function of b_i^*, the assumption that $b_H^* > b_i^* > b_L^*$ requires $b_H^* = \frac{V_i}{|\mathcal{B}^A| - |\mathcal{B}^L|}$. This implies $E\Pi_i(b^*) < 0$.
Step 3: $|\mathcal{B}^A| \leq 2$.
Take $B_i \in \mathcal{B}^L$. For $b_i \leq b_L^*$ we have $E\Pi_i(b_i, b_{-i}^*) = b_i\left(\frac{V_i}{|\mathcal{B}^A|b_H^*} - 1\right)$. Note that b_L^* must maximize this expression independently of whether $b_L^* = b_H^*$ or not. Hence, $b_L^* > 0$ implies $b_H^* \leq \frac{V_i}{|\mathcal{B}^A|}$. Note that if $b_L^* < b_H^*$, then $b_H^* = \frac{V_i}{|\mathcal{B}^A|}$ and $E\Pi_i(b_i, b_{-i}^*) = 0$. For

$b_i \geq b_H^*$ we have

$$E\Pi_i(b_i, b_{-i}^*) \geq V_i \left(1 - \frac{b_H^*}{b_i} \frac{|\mathcal{B}^A| - 1}{|\mathcal{B}^A|}\right) - b_i \geq V_i \left(1 - \frac{V_i}{b_i} \frac{|\mathcal{B}^A| - 1}{|\mathcal{B}^A|^2}\right) - b_i, \tag{A.8}$$

where the first inequality comes from the fact that more than one contestant might bid b_L^* and the second from the bound on b_H^*. The maximizer of the last expression in (A.8) is $b_i^{**} = \frac{\sqrt{|\mathcal{B}^A|-1}}{|\mathcal{B}^A|} V_i$. Suppose $b_L^* < b_H^*$. If $|\mathcal{B}^A| > 2$ the last expression in (A.8) at b_i^{**} is strictly positive and B_i has a profitable deviation from b_L^*. Suppose $b_L^* = b_H^*$. This requires $b_i^{**} = b_H^*$, implying that $b_H^* = \frac{V_i}{|\mathcal{B}^A|}$ and $|\mathcal{B}^A| = 2$ must hold.
The second part of the statement follows, because Theorem 3.1 applies, given that in any equilibrium all but exactly two contestants abstain. *Q.E.D.*

A.7. Proof of Theorem 4.4

Consider the mixed strategy equilibrium of Theorem 3.3. Assume there are further bidders B_j with $j > 2$ who have valuations lower or equal to V_2 and that such a bidder considers to bid any pure strategy $b' > 0$. Given μ_1^*, contestant B_2 obtains $E\Pi_2(\mu_1^*, b') \leq 0$ in the two contestants game. Since the serial contest is anonymous and $V_2 \geq V_j$, we have $0 \geq E\Pi_2(\mu_1^*, b') = E\Pi_2(\mu_1^*, b', 0) \geq E\Pi_j(\mu_1^*, 0, b') > E\Pi_j(\mu_1^*, \mu_2^*, b')$. The strict inequality comes from the fact that the serial CSF assigns for each event in which $b_2 > 0$ in the support of μ_2^* a strictly lower win probability to both B_1 and B_j than when $b_2 = 0$. *Q.E.D.*

A.8. Proof of Corollary 5.1

For $\alpha \leq 1$ the result follows from Theorem 3.4. For $\alpha \geq 1$ consider again the finite approximation with normalization $\hat{V} = 1$. In a symmetric equilibrium $D = 2E(\tilde{\mu}_i^G)$. Given $E(\tilde{\mu}_i^G) = \frac{1}{2} - E\Pi_i(\tilde{\mu}^G)$ and $0 \leq E\Pi_i(\tilde{\mu}^G) \leq \frac{1}{G}$, we have that $1 \geq D \geq 1 - \frac{2}{G}$. *Q.E.D.*

A.9. Proof of Corollary 5.4

Let $n = 2$, $V_1 > V_2$ and m be in the range specified by (5.1). Notice that the upper bound implies that, whatever the value of α, the cap restricts at least B_1's optimal bid. We show first that there exists a pure strategy equilibrium in which both contestants bid m. Suppose contestant B_i bids $b_i = m$. We have that for $B_j \neq B_i$, $E\Pi_j(b_j, m) = \frac{1}{2}\left(\frac{b_j}{m}\right)^\alpha V_j - b_j$ holds. Let $\alpha = 1$ and note that, since $m < V_2/2 \leq V_j/2$, $E\Pi_j(b_j, m)$ is strictly increasing. Thus, $b_i = m$ is optimal. Consider $\alpha < 1$. $E\Pi_j(b_j, m)$ is a strictly concave function. Moreover, it is strictly increasing at $b_j = m$ if and only if

$m < (\alpha V_j)/2$. This holds because of (5.1) and $V_1 > V_2$. Let $\alpha > 1$. $E\Pi_j(b_j, m)$ is a strictly convex function which is strictly decreasing at $b_j = 0$ and strictly increasing at $b_j = m$. In addition at $b_j = m$ we have that $E\Pi_j(m, m) = V_2/2 - m > 0$, because of (5.1). Consider expected total effort. It is increased if and only if $ETE < 2m$, where ETE is specified in Theorems 3.3 and 3.4. This yields the lower bound in (5.1). $Q.E.D.$