A discusión

FUNCTIONAL SUNSPOT EQUILIRIA

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ABSTRACT

Consider a one step forward looking model where agents believe that the equilibrium values of the state variable are determined by a function whose domain is the current value of the state variable and whose range is the value for the subsequent period. An agent's forecast for the subsequent period uses the belief, where the function that is chosen is allowed to depend on the current realization of an extrinsic random process, and is made with knowledge of the past values of the state variable but not the current value. The paper provides (and characterizes) the conditions for the existence of sunspot equilibria for the model described.

Keywords: extrinsic uncertainty, stochastic equilibria

1. Introduction

This paper is about the existence of sunspot equilibria, i.e. stochastic equilibria with self-fulfilling expectations driven purely by extrinsic uncertainty. We specify a class of stochastic equilibria in a framework where agents' beliefs about the dynamic law of the system are in the form of functions that map the value of the state variable at a date to its value at the subsequent date.[3] Agents believe that there is perfect correlation between the sunspot, the realization of a stationary finite state Markov process that models extrinsic uncertainty, and the function that determines the dynamic law to be used. They form their expectations using that function together with the value of the state variable realized in the preceeding period, and that expectation determines the market clearing value of the state variable at the current date. Expectations are self-fulfilling since, in every sunspot state, the actual function linking the state variable across two successive periods is required to coincide with the agents' believed function.

We give the name *functional sunspot equilibria* (henceforth, FSE) to these Markov equilibria. FSE are the stochastic counterparts of nonstationary perfect foresight paths; they exist quite generally and lead to very rich stochastic dynamic behaviour with a parsimonious parameterization. So our formulation, wherein agents form forecasts based on a systematic relationship between past and present prices, i.e. a "theory", with the proviso that their beliefs about the functioning of the economy may be affected by extrinsic uncertainty, has important implications for the existence of equilbria and adds to the scope for multiplicity of rational expectations equilibria already identified in the literature.

We concentrate on the specific case of nonlinear one step forward looking self-referential models with a one dimensional state variable, a specification which includes the basic Overlapping Generations model (henceforth, OLG) with two period lived agents, one consumption good and a constant positive stock of money. We provide an existence result for FSE after which we examine them in greater detail in the basic OLG model, the canonical model in which, traditionally, sunspot equilibria have been analyzed. We find that a necessary and sufficient condition for the existence of FSE is that the Markov transition matrix be singular and that there exist a set of values for the state variable in which the perfect foresight dynamics are well defined and the set is invariant in the "forward" dynamics. When the existence result is applied to the OLG case it shows that FSE exist for *all* parameter configurations; in particular, we obtain existence in the gross substitutes case.

[3]The sunspot equilibria for a linear OLG model discussed in Shell (1977) had a similar feature.

A key element in the notion of FSE is that an agent's expectation, held at a date t, about the value of the state variable in the next period, $t+1$, is formed prior to the realization of the state variable at t. Our motivation for such a specification comes from the fact that in many markets agents are required to act (place demands) before the actual prices at which trade is carried out are available even though the market clearing price is determined on the basis of the expressed demands.[4] The simplest way to close such a model is to postulate that agents beliefs about the functioning of the economy are summarized by a functional relationship between past and current prices, i.e. "backward looking theories" (which can be called "first order theories" since they take into account the price in the previous period), and that they form their beliefs by using the theory corresponding to the current realization of the sunspot and the value of the state variable at the preceeding date. Since "theories" are required to be self-fulfilling, in our approach the axiom of self-fulfilling expectations takes the form of correct "first order theories" that agents have, i.e. "theories" about the process generating the dynamics of the economy.

Much of the existing literature on sunspot equilibria in dynamic economic models provides results on a narrow class of equilibria that have been called finite state stationary sunspot equilibria (henceforth, finite SSE). These are equilibria in which the agents' form expectations according to the belief (or "theory") that there is perfect correlation between the current realization of the sunspot, which follows a stationary finite state Markov process, and the value of the state variable; implictly, one assumes that the accompanying information structure is one where, before going to the market at a date, agents observe the sunspot realization at that date and this pins down the next period's price distribution since the Markov transition probabilities are known. The axiom of self-fulfilling expectations now requires that the "zero order theory" held by the agents be validated.

Most studies on finite SSE are in the framework of the basic OLG model.[5] It is well known that the existence of finite SSE is intimately tied to the existence of a certain kind of invariant set in the perfect foresight dynamics of the deterministic model, a condition that can be related to the indeterminacy of the steady state;[6] in particular, in the basic OLG model with a positive stock of money, finite SSE do

[4]E.g. trading in mutual funds provides an instance where the actual trade often takes place at prices quoted at the end of the trading day after the agents have placed their demands.

[5]See, e.g. Azariadis (1981), Azariadis and Guesnerie (1986), Grandmont (1986), Guesnerie (1986), and Peck (1988).

[6]The precise sufficient condition for the existence of finite SSE and its generalization to an arbitrary number of states is that the invariant set include the steady state in its relative interior, a property that is guaranteed when the steady state is indeterminate.

not exist when demand functions have the gross substitutes property.[7]

In comparing FSE and finite SSE, we observe first that in both agents hold "theories" regarding the precise manner in which the extrinsic process affects the functioning of the economy and the "theories" are self-fulfilling. The principal difference lies in the degree of complexity of the "theories"—finite SSE are sustained by a simpler "zero order theory" where the value of the state variable tomorrow depends only on the sunspot realization today (through the Markov transition probabilities), whereas FSE are "first order theories" that require not only the current realization of the sunspot but the dynamic law that goes with it, hence the preceeding value of the state variable, be taken into account.[8] Secondly, we note that the latter appear as special cases of our formulation where the sunspot dependent belief functions reduce to statewise constants; naturally, conditions under which finite SSE exist are more demanding than those for FSE. FSE exist much more generally and, in the absence of extrinsic uncertainty, FSE coincide with possibly nonstationary perfect foresight solutions of the model (while finite SSE coincide with stationary solutions). We believe that the conceptual innovation of explicitly considering "first order theories" is important (we noted the implications for existence etc. earlier) since most of the literature on sunspot equilibria has concentrated on the case of "zero order theories" that sustain finite SSE.

We stress that even though FSE exist more generally and display substantially richer stochastic dynamic behaviour we must exercise caution since in those cases where FSE exist but finite SSE do not, e.g. the gross substitutes case of the basic OLG model, we are unable to show the existence of permanent oscillations with a uniform lower bound on the amplitude, i.e. the FSE trajectories are random but appear to converge to a constant value with probability one. So the richer dynamic possibilities shown by FSE are best thought of as short term fluctuations. Also, since the indeterminacy of a steady state implies the existence of an invariant set in the forward perfect foresight dynamics, our analysis provides a direct route to the conjecture that if the steady state is indeterminate then simple sunspot equilibria exist.[9] In effect, we provide a generalization of finite SSE, interpretable as going

[7]Even in the gross substitutes case, the steady state is indeterminate if it is supported by a "negative" amount of money, i.e. debt, a case that has traditionally received scant attention in the literature. Davila (1994) provides an existence result for finite SSE that applies in such cases.

[8]In order to obtain well defined sequences of temporary equilibria with self-fulfilling expectations in situations where agents use a functional representation to form expectations as is the case in this paper, it is essential that the belief functions be backward looking. We refer the reader to Grandmont and Laroque (1991) for an elaboration of this more general point.

[9]In more general models, finite SSE need not exist even though the steady state is indeterminate, e.g. Davila (1997) with a predetermined variable. The existance of FSE in such a model is an

from a "zero order theory" to a "first order theory", and, as a consequence, we show that the problem of multiplicity of rational expectations equilibria may be considerably more severe than believed based on the analysis of finite SSE.

Evidently, one can think of other definitions of sunspot equilibrium.[10] Most of these alternatives have had less impact than finite SSE probably because of the simplicity of finite SSE. One such alternative that leads to rich trajectories has been considered by Chiappori and Guesnerie (1993) under the name "random walk" equilibria where the state variable follows a random walk on a countable state space and the limits of the trajectories are steady states of the model. Most of their analysis is geared towards the basic OLG model and they show that such equilibria can exist in the gross substitutes case.[11] It is worth noting that FSE provides a simpler way of obtaining behaviour akin to the random walk equilibria of Chiappori and Guesnerie (1993) though the two existence results are quite different at first sight.[12]

Finally, we remark that in work done independently of the research reported in this paper, Magill and Quinzii (2003) propose an equilibrium concept that, in the special case of extrinsic uncertainty, coincides with FSE.[13] Our focus is different from theirs and so is our existence result since it brings out the importance of the singularity of the transition matrix when uncertainty is extrinsic; when comparing

interesting question.

[10]Woodford (1986) studies sunspot equilibria in general models with many goods with predetermined variables. He considers the case where equilibria are based on a sunspot process that can be any stationary stochastic process and focuses on the relation between determinacy of the steady state and the existence of SSE. Bloise (2004) is a recent contribution in the same vein. Woodford (1994) considers equilibria that need not even be stationary. In all of the above agents hold "zero order theories" though the sunspot process can be quite complicated.

In the case of the basic OLG model, Grandmont (1986) provides a generalization of finite SSE to Markov processes with a compact state space and strictly positive and continuous transitions; the nature of the existence result, however, remains the same as in the finite state case.

[11]In this case, as with FSE, it is believed that the trajectories converge to a constant value with probability one though this issue has not been discussed in the literature.

[12]In addition to requiring that there be two steady states, the existence of random walk equilibria also imposes conditions on the transition probabilities of a random walk money supply where money transfers are proportional. Chiappori and Guesnerie (1993) do not discuss the possibility of obtaining random walk equilibria with a deterministic money supply and sunspot induced random prices. This should be possible but it is not obvious.

[13]They consider a one good OLG model with stochastic endowments and positive money, and the gross substitutes case. They argue in favour of equilibria built around a stationary expectation function and derive the invariance requirement as a necessary condition. They show that such equilibria exist in large numbers and they provide a condition under which all trajectories converge to one or the other strongly stationary equilibrium which are the analogues of the two steady states in the deterministic case. They note that their equilibrium concept also applies to the case of extrinsic uncertainty proving existence even though finite SSE do not exist in their specification due to the gross substitutes property.

our results with the analysis in Magill and Quinzii (2003), one can interpret our result on the role of the singularity of the transition matrix as yet another way in which sunspot equilibria are singular cases.

Section 2 of the paper introduces FSE and provides an existence result. Section 3 relates FSE to finite SSE and other equilibrium concepts of the sunspot type and provides an analysis of FSE in a linear formulation which, because of its simplicity, gives insight into the nature of the more general problem.[14] Section 4 applies the analysis to the OLG setting and Section 5 concludes the paper.

2.1 A Deterministic Model

Consider a deterministic economic model where the state variable is one dimensional. x_t, the equilibrium value of the state variable at date t, is determined given x^e_{t+1}, the value of the state variable expected to prevail the next period, according to a functional relationship:

$$V(x^e_{t+1}) = U(x_t), \tag{1}$$

where $x_t \in X$, the state space with $X \subset R$, and $x^e_{t+1} \in R$. Let $\mathcal{G}(x)$ denote the set of values of x^e that solve (1) for each value of x; in words, $\mathcal{G}(x)$ is the set of beliefs that rationalize the choice x. As x varies in X, a correspondence $\mathcal{G}$ is generated.

In order to proceed further, we need to specify a rule by which x^e_{t+1} is determined. Let us assume that the information structure that agents have access to does not contain the current value of the state variable. Hence, the forecasting scheme employed by agents cannot depend on the current value of the state variable. As discussed in the introduction, this leads us naturally to the use of a forecasting process that is "backward looking."

So suppose that agents hold the theory that the law of motion for the state variable takes the form $x_{t+1} = g(x_t)$ for $x_t \in D$ where $D \subset X$. In our framework, this induces beliefs about future values of the state variable according to $x^e_{t+1} := g(x_t)$ for $x_t \in D$. Since x_t is not known when the forecast is made, one has

$$x^e_{t+1} = g(g(x_{t-1})). \tag{2}$$

We would like the beliefs of the agents to be "consistent", i.e. correctly specified. Since the agent uses (2) to form her forecast, and the true value of x_t is determined according to (1), the consistency condition is modelled by requiring that the believed law of motion be validated, that is, $x_t = g(x_{t-1})$ for $x_{t-1} \in D$, so that

$$V(g(g(x_{t-1}))) = U(g(x_{t-1})) \text{ for all } x_{t-1} \in D.$$

Definition 1: The first order theory given by the function g is *self-fulfilling* on D if

[14] Chiappori, Geoffard and Guesnerie (1991) have studied finite SSE in a linear framework.

$V(g(g(x))) = U(g(x))$ for all $x \in D$.

Evidently, the requirement that g be self-fulfilling is met if $g(x) \in D$ for all $x \in D$ and g is a selection from $\mathcal{G}$. Conversely, if g is self-fulfilling then the fact that $U(g(x))$ must be well defined implies that $g(x) \in X$ for all $x \in D$, while the fact that $V(g(g(x)))$ is well defined implies that $g(g(x))$ is well defined so that $g(x) \in D$ must also hold, and, in addition, g must be a selection from $\mathcal{G}$.

Definition 2: D is *invariant* for g if $\quad x \in D \;\Rightarrow\; g(x) \in D$.

We have shown that g is self-fulfilling on D if and only if D and g are such that

$$x \in D \;\Rightarrow\; g(x) \in D \quad \text{and} \quad V(g(x)) = U(x) \text{ for all } x \in D, \tag{3}$$

so that D is invariant for g and g is a selection from $\mathcal{G}$.

2.2 Functional Sunspot Equilibria

With the groundwork in the deterministic case behind us, we can turn to the subject matter of this paper, the stochastic case. We wish to permit the possibility that agents believe that the realization of a random process affects the law of motion of the economy. Specifically, we assume that agents observe a time homogenous finite state Markov process taking values μ_s, $s = 1, \cdots, N$, with transition matrix Π (with typical entry $\pi_{ss'}$ the probability of transiting to state s' next period conditional on being in state s in the current period where $\pi_{ss'} \in [0,1]$ and $\sum_{s'=1}^{N} \pi_{ss'} = 1$). Without loss of generality, we set $\mu_s = s$.

Consider the information structure wherein, at date t, agents have information upto date $t-1$ on the state variable and *also know* the realization of the sunspot at date t, denoted s_t. They believe that the law of motion for the state variable takes the form $x_{t+1} = h(x_t; s_{t+1})$ so that the realization s_{t+1} affects the function taking x_t to x_{t+1}. The beliefs about future values of the state variable are now induced according to $x^e_{t+1} = h(x_t; s_{t+1})$ for $x_t \in D$. The method for forecasting is the natural extension of that used in the deterministic case as described in (2): at date t, agents forecast x_{t+1} *before* they observe x_t and s_{t+1} and so the forecast is given by a random variable conditional on x_{t-1} and s_t

$$x^e_{t+1} = h(h(x_{t-1}; s_t); s_{t+1}) \text{ for } x_t \in D. \tag{4}$$

We wish to impose a natural consistency requirement on the beliefs of agents, namely, that the state dependent belief functions be self-fulfilling. The requirement is that when (4) is used to form forecasts, and when x_t is determined by solving

$$\sum_{s_{t+1}=1}^{N} \pi_{s_t s_{t+1}} V(h(h(x_{t-1}; s_t); s_{t+1})) = U(x_t),$$

the stochastic analogue of (1), then $x_t = h(x_{t-1}; s_t)$ is validated for all $x_t \in D$ and every state $s = 1, \cdots, N$. This leads us to say that

Definition 3: The stochastic first order theory given by the functions $h(\cdot; s)$, $s = 1, \cdots, N$, is *self-fulfilling* on D if

$\sum_{s'=1}^{N} \pi_{ss'} V(h(h(x; s); s')) = U(h(x; s))$, for all $x \in D$ and for all $s = 1, \cdots, N$.

As in the deterministic case, the functions $h(\cdot; s)$, $s = 1, \cdots, N$, are self-fulfilling on D in the stochastic case if and only if

$$x \in D \Rightarrow h(x; s) \in D \quad \text{and} \quad \sum_{s'=1}^{N} \pi_{ss'} V(h(x; s')) = U(x), \;\; s = 1, \cdots, N. \quad (5)$$

It is obvious that (5) can have trivial solutions in which uncertainty plays no role.

Definition 4: The stochastic first order theory $h(x; s)$, $s = 1, \cdots, N$, leads to a *nondegenerate solution* if

for some $x \in D$ and some $s = 1, \cdots, N$, $\quad V(h(x; s)) - U(x) \neq 0$.

A *finite state functional sunspot equilibrium (FSE)* is a nondegenerate solution of the system (5).

Remark 1: In reading (5) one notices that the description given is time invariant. However, when we construct trajectories that satisfy (5), time subscripts necessarily enter and the correct way of specifying trajectories is by using the rule $x_t = h(x_{t-1}; s_t)$. One could consider an "alternative" specification in which the agents' theories take the form $x_{t+1} = h(x_t; s_t)$ in which case instead of (5) we get the condition

$$x \in D \Rightarrow h(x; s) \in D \text{ and } \sum_{s'=1}^{N} \pi_{ss'} V(h(x; s)) = U(x), \quad s = 1, \cdots, N,$$

an equation that no nontrivial stochastic solution.[15] Our specification of the timing is the only one that permits interesting stochastic solutions.

2.3 Existence

With our equilibrium notion in place, we can state and prove an existence result. We provide a constructive proof to show that if we have a pair D and g where D is invariant for g, g is a selection from $\mathcal{G}$, and the image of D under the composition of V with g is a nondegenerate interval, then FSE exist whenever the transition matrix Π is singular. Clearly, the conditions require that D be a nondegenerate interval.

Proposition 1: *Assume that there exists a pair D and g such that $x \in D \Rightarrow$*

[15] In an earlier version of the paper we had erroneously claimed that, with the alternative specification, the only nontrivial solutions are randomizations over the set $\mathcal{G}(x)$. We thank the referee for bringing the stronger implication to our attention.

$g(x) \in D$ and $V(g(x)) = U(x)$ for all $x \in D$, and that $V(g(D))$, the image of D under the composition of V with g, is a nondegenerate interval. Then, for any $N > 1$ and any Π which is singular, an FSE exists.

Proof: Let $Z := \{z \in R^N : \Pi \cdot z = 0_N, \| z \| = 1\}$, where 0_N is a column vector of 0s. Since Π is singular, $Z \neq \emptyset$. Consider $\tilde{z} \in Z$ such that all the coordinates take the same value; since $\sum_{s'=1}^{N} \pi_{ss'} = 1$ for all $s = 1, \cdots, N$, we must have $\Pi \cdot \tilde{z} = \tilde{z}$. So $\tilde{z} = 0_N$ since $\tilde{z} \in Z$ by hypothesis and $\Pi \cdot z = 0_N$ for $z \in Z$ and that is a contradiction since, necessarily, $0_N \notin Z$. It follows that if $z \in Z$, then $z_i \neq z_j$ for some i, j. Furthermore, $\text{Max}\{z_1, \cdots, z_N\} > 0$ and $\text{Min}\{z_1, \cdots, z_N\} < 0$ so that for any $\xi : D \to Z$, $\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\} > 0$ and $\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\} < 0$.

Set $\overline{V}_D := \sup_{x \in D} V(g(x))$, $\underline{V}_D := \inf_{x \in D} V(g(x))$, and $\widetilde{D} := \{x \in D : V(g(x)) \in (\underline{V}_D, \overline{V}_D)\}$. $\widetilde{D} \neq \emptyset$ since, by hypothesis, $(\underline{V}_D, \overline{V}_D) \neq \emptyset$. It follows that for any $\xi : D \to Z$,

$$0 \in \left(\frac{\underline{V}_D - V(g(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}}, \frac{\overline{V}_D - V(g(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}} \right) \quad \text{for all } x \in \widetilde{D}$$

$$0 \in \left(\frac{\overline{V}_D - V(g(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}}, \frac{\underline{V}_D - V(g(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}} \right) \quad \text{for all } x \in \widetilde{D}.$$

Hence, for any $\xi : D \to Z$ one can induce a function $\alpha : D \to R$ such that

$\alpha(x) \neq 0$ for all $x \in \widetilde{D}$ and

$$\alpha(x) \in \left(\frac{\underline{V}_D - V(g(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}}, \frac{\overline{V}_D - V(g(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}} \right) \cap \left(\frac{\overline{V}_D - V(g(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}}, \frac{\underline{V}_D - V(g(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}} \right),$$

$\alpha(x) = 0$ for all $x \in D / \widetilde{D}$.

Evidently, since $\xi(x) \in Z$ and $\alpha(x) \in R$, $\Pi \cdot \alpha(x)\xi(x) = 0_N$.

By construction,

$$\overline{V}_D > \alpha(x)\xi_s(x) + V(g(x)) > \underline{V}_D \quad \text{for all } x \in \widetilde{D} \text{ and } s = 1, 2, \cdots, N,$$

and at least one of the inequalities is strict if $x \in D / \widetilde{D}$.

Since $(\underline{V}_D, \overline{V}_D) \subset V(g(D))$, for every $y \in (\underline{V}_D, \overline{V}_D)$ there exists $x \in D$ such that $V(g(x)) = y$; denote by $V_D^{-1}(y)$ such a value of $g(x)$. Define the belief functions

$$h(x; s) := V_D^{-1}\big(\alpha(x)\xi_s(x) + V(g(x))\big) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N.$$

By construction, the functions $h(\cdot; s)$ are all well defined and for every $s = 1, \cdots, N$, $h(x; s) \in D$ for all $x \in D$ so that the invariance condition holds. Also, since

$$V(h(x; s)) = \alpha(x)\xi_s(x) + V(g(x)) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N,$$

and $\Pi \cdot \alpha(x)\xi(x) = 0_N$, it follows that

$$\sum_{j=s'}^{N} \pi_{ss'} V(h(x; s')) = \sum_{j=s'}^{N} \pi_{ss'} \alpha(x)\xi_{s'}(x) + \big(\sum_{j=s'}^{N} \pi_{ss'}\big) V(g(x))$$
$$= V(g(x)) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N. \qquad (6)$$

Since we started with a pair D and g where D is invariant for g and g is a selection from $\mathcal{G}$, (3) holds, which, when combined with (6) leads to

$$\sum_{j=s'}^{N} \pi_{ss'} V(h(x; s')) = U(x) \quad \text{for all } x \in D \text{ and } s = 1, \cdots, N.$$

We have shown that (5) holds for the functions $h(\cdot; s)$ as defined. The property $\alpha(x) \neq 0$ for all $x \in \widetilde{D}$ guarantees that the solution we have constructed is non-degenerate. This completes the proof of the proposition.

Remark 2: The construction in the proof above shows that FSE exist in abundance since there is a lot of flexibility in specifying the functions ξ and α. It is possible to extend the construction of the proof to cases where $V(g(D))$ is not an interval by specifying the functions ξ and α appropriately. Also, it is worth noting that differentiability assumptions play no role in our existence result.

We turn to a result which is in the nature of a necessary condition that is implied by the existence of an FSE. We do not emphasize the invariance properties that the forecasting functions must satisfy since these have been observed for related concepts in earlier literature (see, e.g. Grandmont (1986)); instead we focus on what is new about FSE.

Proposition 2: *If an FSE exists for* Π *then* Π *must be singular.*

The proof of Proposition 2 is trivial: since an FSE is a nondegenerate solution of the system (5), and $\sum_{s'=1}^{N} \pi_{ss'} = 1$, we must have $Z \neq \emptyset$ where $Z := \{z \in R^N : \Pi \cdot z = 0_N, \| z \| = 1\}$. It follows that Π is singular.

It is straight forward to show the following implication of Proposition 2.

Corollary 1: *If an FSE exists for* Π *then there exists an FSE for the same economy and* $\tilde{\Pi}$ *where the rows of* $\tilde{\Pi}$ *are identical; in particular, if* $N = 2$ *then the rows of* Π *are identical. One can therefore always sustain extrinsic uncertainty via an independently and identically distributed process.*

3.1 FSE in relation to finite SSE

The most general notion of a sunspot equilibrium that one can consider is that of a stochastic process for x driven by present and past realizations of extrinsic uncertainty which satisfies an equation system that can be derived from (1). That, of course, is too general to tell us much unless we look at the very special case of low dimensional linear models. Therefore, particular emphasis has been placed on sunspot models in which the induced process for x is sufficiently simple which usually means, as noted in the introduction, that finite SSE are considered so that the sunspot process is taken to be a time homogeneous Markov process with a finite number of states and the realizations of x depend only on the current realization of the sunspot. More formally, a finite SSE for Π is a vector $(x_1, \cdots, x_N)$ such that

$\sum_{j=s'}^{N} \pi_{ss'} V(x_{s'}) = U(x_s), \qquad s = 1, \cdots, N.$

By comparing the above with (5) it is evident that finite SSE and FSE are very different objects even though, at a formal level, it can be seen from Definition 3 that finite SSE can be obtained as a special case of FSE by restricting h to take the form $h(x; s) = x_s$, a fixed value independent of x.

The formulation of FSE, and their existence under the fairly general conditions as established by Proposition 1, derives from the requirement that the agents are backward looking and at date t predict x_{t+1} on the basis of x_{t-1} and the sunspot variable. This corresponds to analysing a case where agents forecasts are not time invariant statewise constants but functions that map past states into future ones, i.e. they hold "first order theories", while finite SSE can be thought of as based on "zero order theories". The motivation for the difference in the formulations can be traced to the difference in the information structure that lies behind each of them: in the case of FSE we assume that information on the value of the state variable is *not* available contemporaneously so that the rule for forecasting x^e_{t+1} cannot depend on the realized value of x_t whereas in the case of finite SSE, knowing today's sunspot state is equivalent to knowing the realized value of x_t. The difference in the concepts shows up as a difference in the conditions for existence; in the case of finite SSE, one needs an invariant set whose existence is guaranteed under very special circumstances (we postpone the details to the next section since most of the results for finite SSE were obtained within the framework of the OLG model).

3.2 The Linear Case

This section is of pedagogic value since we consider the special case that arises when the function that links the current value of the state variable with its forecasted value is linear.

It is convenient to work with a function in deviations of the state variable from its unique steady state value; x denotes the value of the deviation. The function is accordingly expressed as

$$x_t = a x^e_{t+1} \tag{7}$$

where 0 is the unique steady state of the system. Set $k := a^{-1}$; k determines the stability of the perfect foresight dynamics associated with the system (7).

In the deterministic case, a typical agent's beliefs will be assumed to be described by a function $g(x) = \beta x$. Since the information available to the agent at date t includes all realizations of the state variable up to and including $t-1$, the agent iterates twice on the belief g to generate the forecast

$x^e_{t+1}(x_{t-1}) = \beta^2 x_{t-1}.$ (8)

By combining (7) and (8), the actual dynamics of the system are obtained

$x_t = a\beta^2 x_{t-1}.$

In this case *self-fulfilling* belief functions are the fixed points, in β, of the function $\Omega(\beta) = a\beta^2$. Ω has two roots, $\bar{\beta}_1 = 0, \bar{\beta}_2 = \frac{1}{a}$ which are the two self-fulfilling belief functions.

With extrinsic uncertainty described by a Markov chain, the belief functions take the form $h(x_{t-1}; s) = \beta_s x_{t-1}$ and one obtains the following forecasting rule given that $s_t = s$

$x^e_{t+1}(x_{t-1}; s) = \left[\sum_{j=s'}^N \pi_{ss'}\beta_{s'}\beta_s\right] x_{t-1}, \qquad s = 1, \cdots, N.$ (9)

The natural consistency requirement that the state dependent belief functions be self-fulfilling takes a very simple form obtained by combining (9) with (7):

$\beta_s x_{t-1} = a\left[\sum_{j=s'}^N \pi_{ss'}\beta_{s'}\beta_s\right] x_{t-1}, \qquad s = 1, \cdots, N,$

which, because of the linear framework, can be expressed solely in terms of the beliefs β_s, $s = 1, \cdots, N$, as

$k = \left[\sum_{j=s'}^N \pi_{ss'}\beta_{s'}\right], \qquad s = 1, \cdots, N.$

Since $\sum_{j=s'}^N \pi_{ss'} = 1$ for all $s = 1, \cdots, N$, it follows that the system of equations which we wish to solve is

$\Pi \cdot [k1_N - \boldsymbol{\beta}] = 0_N,$ (10)

where, 1_N is a column vector of 1s, and $\boldsymbol{\beta}$ is the column vector of the β_s, $s = 1, \cdots, N$ values.

Propositions 1 and 2 apply since R is an invariant set and $k \neq 0$.

To see how FSE can be constructed, suppose that Π is singular. As in the proof of Proposition 1, there exists a vector $z \in R^N$, $z \neq 0_N$, such that $\Pi \cdot z = 0_N$ and $z_i \neq z_j$ for some i, j. Define $\beta^*_s := z_s + k$. $\boldsymbol{\beta}^*$ solves the system of equations (10) and has the property that $\beta_i \neq \beta_j$ for some i, j.

We can also construct solutions using Π as the variable that one solves for. Trivially, the perfect foresight root $\bar{\beta}_2 = k$ solves (10). Fix any collection of values of β_s, $s = 1, \cdots, N$, $\beta_i \neq \beta_j$, $i, j = 1, \cdots N$, and let $\widetilde{\beta}$ and $\widehat{\beta}$ denote respectively the smallest and largest of these N values. If the perfect foresight root $\bar{\beta}_2$ satisfies $\tilde{\beta} < \bar{\beta}_2 < \widehat{\beta}$, one can solve for $\pi_{ij} \neq 0$, $i, j = 1, \cdots, N$, such that (10) is satisfied.

The usual argument for the existence finite SSE in the linear model is as follows: Given Π, an N state finite SSE is a tuple $(x_1, \cdots, x_N)$ such that

$kx_s = \sum_{s'=1,\cdots,N} \pi_{ss'}x_{s'}, \qquad s = 1, \cdots, N,$

which can also be written as

$[k \cdot I_N - \Pi] \cdot x = 0_N,$ (11)

where I_N is the N-dimensional identity matrix. It is known that solutions exist with non-degenerate probabilities and with $x_i \neq x_j$ for some $i \neq j$ if and only if $|k| < 1$.[16]

By comparing (10) and (11) it should be clear that the conditions for existence of the two kinds of equilibria are very different. As noted earlier, FSE generate finite SSE as special cases when $h(x_{t-1}; s) = x_s$.

4 An Overlapping Generations Formulation

In this section we apply our results to the basic OLG model. We also relate FSE to results on finite SSE and random walk equilibria in the overlapping generations model.

Consider the standard OLG model with two period lived agents, one perishable commodity and a constant stock of fiat money. The utility function of an agent is denoted $u(c_1)+v(c_2)$ where c_1, c_2 are consumption when young and old respectively. Endowments are ω_1, ω_2 respectively. The stock of money is normalized to unity. The following standard assumptions will be made

A.1: *u and v are continuous on $[0, +\infty)$ and twice continuously differentiable on $(0, +\infty)$ with $u' > 0$, $v' > 0$, $\lim_{z\to 0} u'(z) = +\infty$, $\lim_{z\to 0} v'(z) = +\infty$, $u'' < 0$, $v'' < 0$. Furthermore, $\omega_1 > 0$, $\omega_2 > 0$. Finally, $\frac{u'(\omega_1)}{v'(\omega_2)} < 1$ so that we are in the Samuelson case.*

We formulate the dynamics of the model in terms of the level of real balances, i.e. by the inverse of the price of the consumption good with the price of money normalized to one. This normalized price is denoted x.

Denote $V(x) := v'(\omega_2 + x)x$ and $U(x) := u'(\omega_1 - x)x$.

In the absence of sunspot activity in the model, the equilibrium price at date t, x_t, is determined, given x^e_{t+1}, the price that is expected to prevail at date $t+1$, according to the following first order condition:

$$V(x^e_{t+1}) = U(x_t). \qquad (12)$$

As indicated in Section 2, the correspondence $\mathcal{G}$ can be generated by considering, for each x, the values of x^e_{t+1} such that (12) holds. Let $A := \{(x, \mathcal{G}(x))\}$; A is obtained by reflecting the agents' offer curve with respect to the vertical axis. Under A.1, U is a monotone increasing function and both U and V are differentiable. This ensures that A is the graph of a differentiable function with the second coordinate as the independent variable. Furthermore, under A.1, there is a unique $\bar{x} > 0$ such that $(\bar{x}, \bar{x}) \in A$, i.e. a positive steady state. There is a "second" steady state, autarky.

[16] When $N = 2$ this is very easy to check.

By applying Proposition 1, we show that under A.1 FSE *always* exist since the existence of a invariant self-fulfilling belief function is guaranteed. For ease of exposition we consider different classes of economies.

For the first class of economies, we assume that the offer curve is monotone. This requires that V be monotone increasing or that

$V'(x) = v'(\omega_2 + x) + xv''(\omega_2 + x) > 0.$

This leads to the following assumption.

A.2: $1 \geq -\frac{xv''(\omega_2+x)}{v'(\omega_2+x)}$ for all $x > 0$.

It is well known that, under A.1 and A.2, the correspondence $\mathcal{G}$ is single valued, hence a function which we denote g; $D := [0, \bar{x}]$, where $\bar{x}$ is the unique monetary steady state, with g is the required pair which gives an invariant self-fulfilling belief function. Note that if we were to set $D = [0, x]$, with $x > \bar{x}$, then the invariance property is lost; however, $D = [0, x]$ with $x \leq \bar{x}$ *does* satisfy the invariance requirement. Similarly, if we set $D = [a, x]$ for $0 < a < x$, the invariance property is lost.

Corollary 2: (gross substitutes economies) *Let A.1 and A.2 hold. For any* Π *which is singular, FSE exist with* $D := [0, x]$ *with* $x \leq \bar{x}$*, and* g *defined by, for each* $x \in D$*,* $g(x) =: \tilde{x}$ *where* $V(\tilde{x}) = U(x)$*.*

As we pointed out in the introduction, finite SSE do not exist under A.2.

While Corollary 2 establishes the existence of FSE, it does not provide information about the possible shapes of the various $h(\cdot; s)$ functions that may appear as solutions to the system of equations that define FSE. In Example 1 below we specify a particularly simple type of FSE with $N = 2$ where, in the first state, the state variable moves towards the monetary steady state and in the second state, towards the autarkic one. Specifically, we take $D := [0, \bar{x}]$ and $N = 2$ and obtain solutions $h(\cdot; s)$, $s = 1, 2$, with the property that the two functions lie on either side of the 45 degree line through the origin so that under the action of $h(\cdot; 1)$, the state variable is pulled towards the monetary steady state $\bar{x}$, while under the action of $h(\cdot; 2)$, it is pulled towards the autarkic steady state $x = 0$. We obtain these solution functions in a straight forward constructive manner. It should be clear though that these are by no means the only possible specifications. The model admits other shapes of these functions that need not be monotone and hence the dynamic laws may be more complex than those outlined in Example 1.

Example 1: Assume A.1 and A.2. Fix $h(\cdot; 1)$ defined on $D = [0, \bar{x}]$ satisfying

$h(0;1) = 0$, $h(\bar{x};1) = \bar{x}$, $h(x;1) > x$ for all $x \in (0, \bar{x})$, $h(\cdot;1)$ is strictly increasing, and differentiable around 0 (from the right). It follows that $\lim_{x\to 0} h'(x;1) > 0$. g which solves (1) is also increasing, $g(x) \leq x$ for all $x \in [0, \bar{x}]$, and differentiable around 0 (from the right) with $\lim_{x\to 0} g'(x) > 0$ (since we are in the Samuelson case). It follows that $V(g(x)) < V(h(x;1))$ for all $x \in (0, \bar{x})$ since V is increasing under A.2. Choose $\pi \in (0,1)$ such that $\pi V(h(x;1)) \leq V(g(x))$ for all $x \in [0, \bar{x}]$. Clearly, such a π exists since

$$\lim_{x\to 0} \frac{V(g(x))}{V(h(x;1))} = \lim_{x\to 0} \frac{V'(g(x))g'(x)}{V'(h(x;1))h'(x;1)} = \lim_{x\to 0} \frac{g'(x)}{h'(x;1)} > 0.$$

It follows that

$$0 \leq \frac{V(g(x)) - \pi V(h(x;1))}{1-\pi} \leq V(g(x)) \qquad \text{for all } x \in [0, \bar{x}].$$

Set $h(x;2) := V^{-1}\left(\frac{V(g(x)) - \pi V(h(x;1))}{1-\pi}\right)$ for all $x \in [0, \bar{x}]$; by the inequality above, $h(\cdot;2)$ is well defined. It is immediate that $h(0;2) = 0$ and $h(\bar{x};2) = \bar{x}$, since we know that $g(0) = h(0;1) = 0$ and $h(\bar{x}) = h(\bar{x};1) = \bar{x}$. From (6), $h(\cdot;1)$ and $h(\cdot;2)$ constitute an FSE if $\pi V(h(x;1)) + (1-\pi)V(h(x;2)) = V(g(x))$ for all $x \in [0, \bar{x}]$; we have constructed an FSE since $h(\cdot;2)$ was defined so as to satisfy the equation. We now show that $h(x;2) < g(x)$ for all $x \in (0, \bar{x})$. Since g and $h(\cdot;1)$ are increasing functions and $h(x;1) > g(x)$ for all $x \in (0, \bar{x})$ by construction, and V is an increasing function, one has $V(h(x;1)) > V(g(x))$; hence, by (6), $V(h(x;2)) < V(g(x))$ which implies in turn that $h(x;2) < g(x)$ for all $x \in (0, \bar{x})$. Finally, note that the construction imposes a very mild condition on $\lim_{x\to 0} h'(x;1)$; the derivative could exceed one thus allowing nonconvergence to the autarkic steady state.

We turn to the case where A.2 does not hold. The offer curve must bend backwards and there exists $x^* \in (0, \omega_1)$ such that $V'(x) \geq 0$ for all $x \in [0, x^*]$. $\mathcal{G}$ is no longer a function; instead one uses two (or more if necessary) functions, g^+ and g^-, where $g^+(x^*) = g^-(x^*)$. If $g^+(x^*) \geq x^*$ then Corollary 2 goes through unchanged (this is the case in which the offer curve bends "late" and the steady state is on the "lower" branch, $g^+(\bar{x}) = \bar{x}$). If instead $g^+(x^*) < x^*$, so that the offer curve bends "early" and the steady state is on the "upper" branch, $g^-(\bar{x}) = \bar{x}$, then one possibility is to modify the construction by setting $D := [0, x^*]$ and choosing a selection from $\mathcal{G}$ which is invariant—there are various possibilities, e.g. (i) choose g^+, (ii) choose g^+ on $[0, \hat{x}]$ and g^- on $(\hat{x}, x^*]$ where $\hat{x}$ is such that $V(x^*) = U(\hat{x})$.

Corollary 3: (backward bending offer curves) *Let A.1 hold. For any* Π *which is singular, FSE exist with (i)* $D := [0, x]$ *with* $x \leq \bar{x}$*, if* $\bar{x} \leq x^*$*, or with (ii)* $D := [0, x^*]$*. In either case* g *can be specified to make* D *invariant.*

The solutions described in Corollaries 2 and 3 have the property that they cannot be bounded away from the autarkic steady state.[17] This leads us to ask if there are other solutions where the dynamics of the state variable are confined to some neighbourhood of the monetary steady state and bounded away from the autarkic steady state. The answer is yes: An invariant self-fulfilling belief function exists around the monetary steady state if and only if a cycle of period two exists. Here one can set $D := [x_1, x_2]$, where $0 < x_1 < x_2 < x^*$ are the values of the state variable along the two cycle. It is known that under an additional nondegeneracy assumption regarding the two cycle, the stated condition is also necessary and sufficient for the existence of finite SSE under A.1.[18] So FSE can exist when a critical cycle of period two exists, a case where finite SSE need not exist.

Corollary 4: (backward bending offer curves) *Let A.1 hold. For any* Π *which is singular, FSE with the property that the dynamics of the state variable are confined to an interval around the monetary steady state and bounded away from the autarkic steady state exist if and only if a cycle of period two exists.*

We can now discuss the relation of FSE with earlier work on sunspots within the OLG framework. As indicated earlier, the existence of finite SSE requires a particular kind of invariant set for the deterministic dynamics. The easiest way to meet the requirement is to posit that the steady state be indeterminate in the forward dynamics. Examples of such results are Azariadis and Guesnerie (1986), Grandmont (1986) and Guesnerie (1986). Corollary 4 shows that FSE also obtain under the same circumstances and for an additional nongeneric case to boot. However, FSE obtain far more generally as Corollaries 2 and 3 have shown.

The random walk equilibria introduced by Chiappori and Guesnerie (1993) are very different compared to finite SSE and they too can exist in the gross substitutes case. Though they behave like the FSE in Example 1, the conditions for existence of random walk equilibria are somewhat different. More importantly, we feel that the constructive proof of our Proposition 1 together with the additional details provided in Example 1 are much easier to follow. We prefer to view FSE as a much simpler approach to obtaining random walk type behaviour.

4. Concluding Remarks

In this paper we introduced FSE, an equilibrium concept that subsumes finite SSE, and demonstrated conditions under which they exist. We noted the extreme

[17] Many of the sunspot equilibria discussed in Woodford (1994) have the same feature.

[18] See Azariadis and Guesnerie (1986) or Grandmont (1986).

degree of flexibility that one has in constructing FSE since g need not be continuous and the set Z, the null space of Π, can be large. This shows once again that the requirement of self-fulfilling beliefs in itself is very weak and far from letting one pin down the equilibrium behaviour of economic systems.[19] In our analysis, the multiplicity is driven by the fact that the state variable is endogenous and is not tied down sufficiently by the fact that it is described by a function relating its value across successive periods.

We remark that an infinite time horizon is essential to obtaining FSE since the dynamics of the state variable inherently depend on the past value of the state variable through functions that are state dependent and self-fulfilling; it follows that the sunspot equilibria of the canonical two period model (Cass and Shell 1983) cannot be obtained as FSE.

Also, it remains to investigate the welfare properties of FSE. It is known that finite SSE are ex-ante inefficient and their optimality properties under a weaker (conditional) notion of optimality is easily checked using the "unit root property" due to Aiyagari and Peled (1991); in the case of FSE, while it is easy to show that they too are ex-ante inefficient, their welfare properties under a conditional notion of optimality are not clear since they induce nonstationary paths and the criterion for optimality developed by Chattopadhyay and Gottardi (1999) is the appropriate tool.

Finally, further work is required to determine the precise connection between FSE and the general class of sunspot equilibria studied by Woodford (1986), Chiappori and Guesnerie (1993) and Woodford (1994). Given the large degree of multiplicity of FSE, it would also be interesting to investigate whether any of these might be robust in the sense of Goenka and Shell (1997), and which might be stable under adaptive learning rules.

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