COALITON FORMATION AND STABILITY*

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ABSTRACT

This paper studies a class of NTU coalition formation games in which every player's payoff depends only on the members of her coalition. We identify four natural conditions on individuals' preferences and show that, under each condition, stable (core) allocations exists.

Keywords: Coalition Formation, Core, NTU Games, Stability

Journal of Economic Literature: C78, D78.

1. Introduction

In many economic situations, individuals carry out activities by forming coalitions. In this paper we deal with a family of NTU games, namely coalition formation games that are, using the terminology from Drèze and Greenberg [6], hedonic in the sense that every player's payoff depends only on the members of her coalition. In this framework, we define four conditions called Union Responsiveness, Intersection Responsiveness, Singularity and Essentiality. Each one of them guarantees the non-emptiness of the core of the coalition formation games.

Formally, a NTU game can be described by a function which selects, for each non-empty coalition, a set of outcomes that can be reached by the agents belonging to it. Once a coalition is formed, the agents in that coalition are involved in a bargaining process to determine their outcome. Whenever the bargaining processes, one for each possible coalition, are known by the agents before any coalition were formed, we can describe NTU games in a easier and more natural way. That is, defining a function that associates a *single* allocation to each coalition. Henceforth, if agents' preferences are defined over the set of allocations we can describe an ordering for each agent, whose domain is the set of coalitions she might belong to. Therefore, we could assume that each agent's preferences depend on the set of people belonging to the coalition to which she belongs. In that sense, the resulting problem becomes a pure *hedonic preferences model*.

In this framework, Banerjee et al. [3] have shown that there are no core-wise stable coalition partition even when strong restrictions, such as additive separability or anonymity, are imposed on individuals' preferences even when united with appropriately defined versions of single-peaked preference conditions.

Conditions under which the core is non-empty in hedonic coalition formation games have been studied by several authors. In this sense, Banerjee et al. [3] introduce a condition, they call *the top coalition property*, which is a natural extension of Alcalde's [1] P-reciprocity for the room-mate problem. These authors show that, if agents' preferences satisfy the top coalition property, or a weaker version of it, the core of the coalition formation problem is non-empty.

Bogomolnaia and Jackson [5] also focus on conditions under which a coalition formation problem has stable allocations. These authors identify two properties, they call *ordinal balanceness* and *weak consecutiveness*, and show that when agents' preferences satisfy any of the two conditions, the core of the related coalition formation game is non-empty.

The conditions introduced by Banerjee et al. [3] and by Bogomolnaia and Jackson [5] do not impose restrictions on each agents' preferences, but on the preferences profile, i.e. preferences' domains cannot be expressed as a Cartesian product of agents' preferences. For this reason, the analysis of whether some of these conditions are satisfied or not by agents' preferences becomes a problem that can be as difficult to solve as the (direct) study of the existence of core allocations.

The conditions proposed in this paper are described in each agent's preferences, rather than in the preferences profiles. This fact produces two interesting features. First, it is very easy to evaluate whether or not agents' preferences satisfy our conditions. Hence, it would be possible to design efficient algorithms selecting stable allocations and decide whether these algorithms can be used in a given preferences profile. Second, when conditions are stated on each agent's preferences, it is easy to study whether the introduction of a new agent into the problem introduces instabilities. Note that, if conditions are stated on agents' preference profiles, rather than on individuals' preferences, the analysis of this problem is not limited to the new agent but to the complete profile. This fact will greatly hinder efforts to carry out any comparative static analysis over a certain profile.

Let us conclude this Introduction by briefly describing the conditions introduced in this paper: Union Responsiveness, Intersection Responsiveness, Singularity and Essentiality.

The Union Responsiveness Condition (URC) is a very monotonic condition: From each agent's point of view, "more is better" except if she is in her top. If every agent's preferences satisfy this simple idea we can guarantee the existence of core allocations in the coalition formation game.

The idea behind the Intersection Responsiveness Condition (IRC) is the following: given two coalitions with a non-empty intersection the coalition formed by the agents on the intersection is better than the worse of the two previous coalitions. We provide a non-constructive proof of the existence of stable allocations when the agents' preference satisfies the IRC.

Singularity reflects the idea of *extreme-minded* agents. These agents' preferences can be illustrated by the sentence: "if I do not obtain what I want, I will not cooperate at all."

The last condition we introduce in this paper is called Essentiality. The condition models economic situations where, for each individual, there is a group of agents that belong to any coalition she considers acceptable. In addition, given two possible coalitions that contain her group of essential agents she prefers the one that contains fewer non-essential members.

The rest of the paper is organized as follows. Section 2 introduces the model, identifies the problem to be analyzed, and presents a formal definition for Union

Responsiveness, Intersection Responsiveness, Singularity and Essentiality, four conditions that guarantee the existence of stable coalition structures. To conclude this section, we prove the independence of the above conditions (Proposition 2.6). The results of the paper can be found in Section 3. Conclusions and open questions are addressed in Section 4. Finally, formal proofs are gathered to the Appendix.

2. The Framework and Main Definitions

Consider a set N of agents, $N = \{1, \ldots, i, \ldots, n\}$, who have to form coalitions. Each agent i is endowed with preferences $\succsim_i$ which can be represented by a linear order over the set $\sum_i := \{\tau \subseteq N : i \in \tau\}$. A *coalition structure* $\mathcal{T} = \{\tau^1, \ldots, \tau^j, \ldots, \tau^t\}$ is a partition of the set of agents. Given a partition $\mathcal{T}$ and an agent i, let τ_i denote the element on $\mathcal{T}$ which i belongs to. Finally, and for notational convenience, let us extend agents' preferences to be defined over the set of coalition structures in the following natural way: $\mathcal{T} \succsim_i \tilde{\mathcal{T}}$ if, and only if, $\tau_i \succsim_i \tilde{\tau}_i$.

Our objective in this paper is to analyze, for each coalition formation problem, $\{N, \succsim\}$, the set of allocations that are expected to hold, taking into account agents' preferences. The core describes when an allocation is considered stable and, therefore, is expected to be the consequence of agents' collective decisions.

Definition 2.1. *Let $\{N, \succsim\}$ be a coalition formation problem, and $\mathcal{T}$ be a coalition structure for such a problem. We say that*

1. *$\mathcal{T}$ is Pareto efficient if there is no coalition structure $\mathcal{T}' \neq \mathcal{T}$, such that $\mathcal{T}' \succ_i \mathcal{T}$, for each agent i for which $\tau_i \neq \tau_i^{'}$.*

2. *$\mathcal{T}$ is stable, or is in the core of $\{N, \succsim\}$, if there is no set of agents S, $\emptyset \neq S \subseteq N$, such that, for each $i \in S$, $S \succ_i \tau_i$.*

3. *A set S of players blocks allocation $\mathcal{T}$ whenever $S \succ_i \tau_i$ for all $i \in S$.*

4. *$\mathcal{T}$ is individually rational* if it is not blocked by any individual, i.e., $\tau_i \succsim_i \{i\}$ *for each agent i.*

Unfortunately, as we said before, the set of stable allocations can be empty for some instances. The usual way to escape the emptiness of the core in some families of cooperative games comes from the analysis of specific environments guaranteeing the existence of stable (in the sense of core) allocations. The interest of such an approach needs some further justification in two particular ways. The first one is the richness of the domain restriction; the second one is mostly related to interpretative questions.

In the remaining of this section we introduce some domain restrictions on agents' preferences that will guarantee the existence of stable coalition structures satisfying the two requirements above, in a sense that we will made precise.

We will say that a set of preferences for agent i, say $\mathcal{P}_i$, is rich if it fulfils the following two conditions:

(a) For each $S \in \sum_i$, there are preferences $\succsim_i \in \mathcal{P}_i$ such that S is the $\succsim_i$-maximal on $\sum_i$; and

(b) for each two set $S, S' \in \sum_i$, such that $S \cap S' \neq \{i\}$, $S \backslash S' \neq \{i\}$, and $S' \backslash S \neq \{i\}$, there are preferences $\succsim_i$ and $\succsim_i'$ belonging to $\mathcal{P}_i$ such that $S \cap S' \succ_i S \succ_i S'$, and $S \cap S' \succ_i' S' \succ_i' S$.

The first condition establishes that any set of agents could be the best group of colleagues for agent i. The second one specifies that the maximal of an agent's preferences does not necessarily determine how the other groups are ordered.

Next we proceed to introduce formal definitions. First of all, let us introduce some additional notation. Given a subset of individuals S, and an agent $i \in S$, $Ch_i(S)$ denotes i's choice on S, i.e. $\succsim_i$ $-$maximal on $2^S \cap \sum_i$.

Definition 2.2. *We say that agent i's preferences $\succsim_i$ satisfy the Union Responsiveness Condition, URC in short, if for each agent i and two coalitions $S, S' \in \sum_i$ such that $S' \subset S$ and $S' \neq Ch_i(S)$, $S \succ_i S'$ holds.*

The fulfillment of the following Intersection Responsiveness Condition by agents' preferences is also sufficient to guarantee the existence of stable allocations. (See Theorem 3.2.)

Definition 2.3. *We say that agent i's preferences $\succsim_i$ satisfy the Intersection Responsiveness Condition, IRC in short, if for any pair of coalitions, S and S' in $\sum_i$, $S \succ_i S'$ implies $S \cap S' \succsim_i S'$.*[1]

Singularity, the third condition we propose, has an interpretation that is dual to URC. Under URC, the grand coalition N is one of the two best options for each agent. On the other hand *Singularity* imposes that the stay-alone option, $\{i\}$, is one of the two best options for agent i.

Definition 2.4. *We say that agent i's preferences $\succsim_i$ satisfy Singularity if for each coalition $S \in \sum_i$*

$$S \succ_i \{i\} \text{ implies } S = Ch_i(N).$$

Finally, we introduce a formal definition of Essentiality.

[1] *Let us remember that we are assuming that $\succsim_i$ is a linear ordering. Therefore, this condition should be read that $S \cap S' \succ_i S'$ whenever $S \cap S' \neq S'$.*

Definition 2.5. *We say that coalition* $\tau_i^e \in \sum_i$ *is essential for* i *if, and only if, agent* i*'s preferences,* $\succsim_i$*, satisfy that*

(i) if $\tau_i^e = \{i\}$*, then* $\{i\} \succ_i S$ *for any* $S \neq \{i\}$*, and*

(ii) if $\tau_i^e \neq \{i\}$*, then*

(a) $\{i\} \succ_i S$ *if, and only if,* S *is not a superset of* τ_i^e*, i.e.* $\tau_i^e \backslash S \neq \emptyset$*, and*

(b) for any two coalitions S *and* S' *in* $\sum_i$*, if* $\tau_i^e \subseteq S \subset S'$ *then* $S \succ_i S'$*.*

We say that agent i*'s preferences satisfy Essentiality whenever a coalition exists which is essential for her. Finally, we say that* i *is self-sufficient if* $\{i\}$ *is essential for her.*

These conditions above defined are *logically independent*, i.e. none of the four conditions is implied by the others, as Proposition 2.6 shows.

Proposition 2.6. *Union Responsiveness, Intersection Responsiveness, Singularity and Essentiality are independent conditions.*

Proof. To prove this statement, we will provide agents' preferences that satisfy some of the above mentioned properties, but do not satisfy the remaining three.

[1] *Preferences satisfying URC and not verifying IRC, nor Singularity, neither Essentiality.*

Let us consider an individual, say 1, which exhibits the following preferences for a three-agent problem.

$$\{1,2,3\} \succ_1 \{1,2\} \succ_1 \{1,3\} \succ_1 \{1\}.$$

In this case, agent 1's preferences satisfy URC. Nevertheless, these preferences do not satisfy IRC because $\{1,2\} \succ_1 \{1,3\}$, $\{1,3\} \succ_1 \{1\}$ and $\{1,2\} \cap \{1,3\} = \{1\}$. Note that IRC should imply that $\{1\} \succ_1 \{1,3\}$. Clearly, these preferences do not satisfy Singularity. Moreover, there is no set of agents who are essential for 1.

[2] *Preferences satisfying IRC and not verifying URC, nor Singularity, neither Essentiality.*

Let us consider an individual, say 1, which exhibits the following preferences for a four-agent problem.

$$\{1,2,4\} \succ_1 \{1,2\} \succ_1 \{1\} \succ_1 \{1,4\} \succ_1 \{1,3,4\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1,2,3,4\}.$$

In this case, agent 1's preferences satisfy IRC. Nevertheless, these preferences do not satisfy URC because $\{1,2\} \succ_1 \{1\}$, $\{1\} \succ_1 \{1,2,3\}$ and $\{1,2\} \subset \{1,2,3\}$.

So, these preferences do not fulfill URC. Clearly, these preferences do not satisfy Singularity. Moreover, there is not set of agents who are essential for 1.

[3] *Preferences satisfying Singularity and not verifying URC, nor IRC, neither Essentiality.*

Let us consider an individual, say 1, which exhibits the following preferences for a four-agent problem.

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,3,4\} \succ_1 \{1,4\} \succ_1 \{1,2,3\} \succ_1 \{1,3\} \succ_1 \{1,2,3,4\}.$$

These preferences satisfy Singularity. Nevertheless, they do not satisfy IRC because $\{1,3,4\} \succ_1 \{1,2,3\}$, and $\{1,2,3\} \succ_1 \{1,3\} = \{1,3,4\} \cap \{1,2,3\}$. On the other hand, it is easy to check that these preferences do not satisfy URC, nor essentiality.

[4] *Preferences satisfying Essentiality and not verifying URC, nor IRC, neither Singularity.*

Let us consider an individual, say 1, which exhibits the following preferences for a four-agent problem.

$$\{1,4\} \succ_1 \{1,3,4\} \succ_1 \{1,2,4\} \succ_1 \{1,2,3,4\} \succ_1 \{1\} \succ_1 \{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1,3\}.$$

In this case, agent 1's preferences satisfy Essentiality. Note that the set $\{1,4\}$ is essential for agent 1. Nevertheless, these preferences do not satisfy IRC because $\{1,3,4\} \succ_1 \{1,2,3\}$, $\{1,2,3\} \succ_1 \{1,3\}$ and $\{1,3\} = \{1,3,4\} \cap \{1,2,3\} \subset \{1,2,3\}$. Note that IRC will imply that $\{1,3\} \succ_1 \{1,2,3\}$. Moreover, these preferences do not fulfill URC. Let us observe that $Ch_1(\{1,2,3,4\}) = \{1,4\} \neq \{1,2,4\}$, and $\{1,2,4\} \subset \{1,2,3,4\}$. Therefore, URC should imply $\{1,2,3,4\} \succ_1 \{1,2,4\}$, which is not the case for the above preferences. Finally, it is easy to see that these preferences do not satisfy Singularity. ■

3. Existence of Stable Coalition Structures

In this section we show how each one of the conditions we have introduced in Section 2 separately guarantees the existence of stable allocations. The main proofs of our results can be found at the Appendix.

Theorem 3.1. *Suppose that preferences $\succsim_i$ of each agent i satisfy the Union Responsiveness Condition. Then there is at least one stable coalition structure $\mathcal{T}$.*

We next deal with the analysis of agents satisfying the Intersection Responsiveness Condition.

Theorem 3.2. *Let* $\{N, \succsim\}$ *be a coalition formation problem whose agents' preferences satisfy the Intersection Responsiveness Condition. Then the set of stable coalition structures is non-empty.*

Next theorem, whose immediate proof is omitted, points out that frameworks whose agents' preferences fullfil the property called Singularity have always a unique stable allocation.

Theorem 3.3. *Let* $\{N, \succsim\}$ *be a coalition formation problem. Assume that each agent's preferences satisfy Singularity. Then, the unique stable coalition configuration for such a problem is* $\mathcal{T}^S$*, where for each agent, say* i*,*

$$\tau_i^S = \begin{cases} Ch_i(N) & \text{if } Ch_j(N) = Ch_i(N) \text{ for all } j \in Ch_i(N) \\ \{i\} & \text{otherwise} \end{cases}$$

Finally the following Theorem presents our results when the Essentiality condition is satisfied.

Theorem 3.4. *Let* $\{N, \succsim\}$ *be a coalition formation problem. Assume that each agent's preferences satisfy Essentiality. Then, there is a stable coalition configuration for such a problem. Moreover, the set of stable allocations is a singleton.*

4. Final Remarks

In this paper we have introduced some natural conditions that appear in social environments; and we have shown that each one of them can guarantee the existence of stable allocations in coalition formation games. These conditions can be interpreted in economic terms and it makes it easier to decide whether they can be applied to a particular economic situation. Moreover, our conditions are imposed on individual preferences. This simplifies the process of checking whether the restrictions are satisfied or not for a given preference profile. This circumstance is particularly interesting if a new agent is added to a profile of preferences that already satisfies one of the conditions because we can check if the new profile does it as well simply by checking the preferences of the new agent.

On technical grounds we have used both constructive and non-constructive techniques to prove our results. The non-constructive proof provided for the case of agents' preferences satisfying the Intersection Responsiveness condition is inspired by the arguments given in Sotomayor [7] for the case of marriage markets.

The fact that the restrictions are imposed on agents' preferences, not on preference profiles, allows us to consider as an extension of our results the possibility of carrying out the analysis of dynamic aspects in our model. Some of them can be stated straightforwardly. A first approach can be taken by assuming that agents can leave a coalition freely, i.e. agents do not sign a contract requiring that all the agents in a coalition, when formed, are committed to it. In such a case, the analysis of stability can be done following a study of comparative statics. Such a study is simple in our case. A second approach, which becomes more difficult to study, can be stated by assuming that, once a coalition is formed, it can accept new agents but no agent can leave that coalition. In such a case, we need to state which procedure a coalition will employ to decide whether or not to accept new agents. This problem can be dealt with by combining our results with those of Barberà et al. [4], that study the problem of how clubs decide the acceptance of new members in a dynamic setting.

A second extension of the analysis is the study of agents' strategic behavior when faced with rules for selecting stable allocations. A first approach can be found in Alcalde and Revilla [2]. These authors study the case in which agents' preferences satisfy Essentiality. They prove the existence of a strategy-proof mechanism in which agents declare their essential coalition and select the (unique) stable allocation.

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APPENDIX

Coalition Formation and URC

This Appendix will provide a positive answer to the existence of stable allocations in problems whose agents exhibit preferences satisfying URC. An interesting feature of the proof that we provide is that it is constructive; i.e., we give a procedure to build stable allocations for any problem in which the Union Responsiveness Condition is fulfilled.

Previous to give a formal proof for Theorem 3.1, we will introduce two properties induced by URC:

- (a) If $\succsim_i$ satisfies URC on $2^N \cap \sum_i$, then for any $S \in \sum_i$, $\succsim_i$ restricted on $2^S \cap \sum_i$ satisfies URC; and
- (b) If $\succsim_i$ satisfies URC on $\sum_i$, then the grand coalition is either the maximal element for i, or the second best element.

Proof of Theorem 3.1.

We prove the result by induction on the number of players in the game. First, let us observe that our result is true for the two-agent case. Now, let us assume that the result be true for all games where $\succsim_i$ satisfy URC and the number of players is less or equal to $n-1$ and consider a game with $|N| = n$. Let us consider the coalition structure $\mathcal{T} = [N]$, in which the society forms the unique coalition. If $\mathcal{T}$ is stable, the desired result follows. Otherwise, there should be a coalition, say S^1 such that for every $i \in S^1$, $S^1 \succ_i N$. It follows that $Ch_i(N) = S^1$ for each $i \in S^1$. By the induction hypothesis, there exist a stable coalition structure $\mathcal{T}'$ for the set of agents $N \backslash S^1$. It is straightforward to see that the coalition structure $(S^1, \mathcal{T}')$ is stable for the game which has player set N. ■

Coalition Formation and IRC

We provide a non-constructive proof of the existence of stable allocations. In fact, our proof has some similarities with that presented in Sotomayor [7] for two-sided matching markets.

We first introduce a property which describes how the concepts of Pareto efficiency and stability are related under IRC. In particular, we will see that Pareto efficiency and an internal stability property together characterize, in this framework, stability.

Proposition 4.1. *Let $\{N, \succsim\}$ be a coalition formation problem, and $\mathcal{T}$ an allocation for this problem. If agents' preferences satisfy the Intersection Responsiveness Condition, then $\mathcal{T}$ is stable for $\{N, \succsim\}$ if, and only if,* *(i)* $\mathcal{T}$ *is Pareto efficient, and*

(ii) For each coalition τ^j *in* $\mathcal{T}$ *and for any set of agents* $\emptyset \neq S \subset \tau^j$*, there is an agent* $i \in S$ *such that* $\tau^j \succ_i S$*.*

Proof. Note that, if either of the two conditions above are not satisfied by $\mathcal{T}$, this allocation is not stable for $\{N, \succsim\}$. If fact, if Pareto efficiency is not satisfied, any allocation $\mathcal{T}'$ Pareto dominating $\mathcal{T}$ will give us a coalition (which is formed in $\mathcal{T}'$ but it is not in $\mathcal{T}$) blocking $\mathcal{T}$. Moreover, if Condition (ii) is not satisfied, there is a coalition, say τ^j, containing a set S such that S blocks $\mathcal{T}$.

On the other hand, let us consider an allocation $\mathcal{T}$ satisfying conditions (i) and (ii) above. Suppose that $\mathcal{T}$ is not stable. Then, a blocking coalition T exist. Thus, for each agent $i \in T$, $T \succ_i \tau_i$. Since $\mathcal{T}$ is Pareto efficient, there is an agent $\hat{\imath} \in T$ such that $\tau_{\hat{\imath}} \cap T \neq \tau_{\hat{\imath}}$. Then, by Definition 2.3, $T \cap \tau_{\hat{\imath}} \succ_h \tau_{\hat{\imath}}$ holds for each agent $h \in T \cap \tau_{\hat{\imath}}$, which contradicts Condition (ii) above. ■

Lemma 4.2. *Let* $\{N, \succsim\}$ *be a coalition formation problem, and* $F(\succsim)$ *be the set of allocations satisfying Condition* (ii) *in Proposition 4.1. If allocation* $\mathcal{T}$ *is Pareto efficient in* $F(\succsim)$ *then it is Pareto efficient for the problem* $\{N, \succsim\}$*.*

Proof. Let us assume that $\mathcal{T}$ is Pareto efficient restricted to $F(\succsim)$ but is not efficient for $\{N, \succsim\}$. Then, there should be an allocation, say $\widetilde{\mathcal{T}}$ which Pareto dominates $\mathcal{T}$, i.e. $\tilde{\tau}_i \succsim_i \tau_i$ for all $i \in N$ and $\tilde{\tau}_{\hat{\imath}} \succ_{\hat{\imath}} \tau_{\hat{\imath}}$ for some agent $\hat{\imath}$. Since $\mathcal{T} \in F(\succsim)$ there is no agent i for which $\tilde{\tau}_i \subset \tau_i$. Therefore, it should be the case that (a) there is an agent i such that $\tilde{\tau}_i \neq \tilde{\tau}_i \cap \tau_i \neq \tau_i$ or (b) $\tilde{\tau}_i \supseteq \tau_i$ for all i.

If case (a) holds, assume that, for agent $\tilde{\imath}$, $\tilde{\tau}_{\tilde{\imath}} \neq \tilde{\tau}_{\tilde{\imath}} \cap \tau_{\tilde{\imath}} \neq \tau_{\tilde{\imath}}$. In such a case, for each agent $i \in \tilde{\tau}_{\tilde{\imath}}$, $\tilde{\tau}_{\tilde{\imath}} \succ_i \tau_i$. Applying Definition 2.3, we have $\tilde{\tau}_{\tilde{\imath}} \cap \tau_{\tilde{\imath}} \succ_i \tau_{\tilde{\imath}}$ for each $i \in \tilde{\tau}_{\tilde{\imath}} \cap \tau_{\tilde{\imath}}$, which contradicts the fact that $\mathcal{T} \in F(\succsim)$.

If case (b) holds, since $\mathcal{T}$ is efficient restricted to $F(\succsim)$, then $\widetilde{\mathcal{T}} \notin F(\succsim)$. Hence, there is an agent $\tilde{\imath}$ and coalition $S \subset \tilde{\tau}_{\tilde{\imath}}$ such that $S \succ_i \tilde{\tau}_{\tilde{\imath}}$ for all $i \in S$. Assume that there is an agent $\hat{h} \in S$ for which $S \cap \tau_{\hat{h}} \neq \tau_{\hat{h}}$ (otherwise, an allocation can be built in $F(\succsim)$ Pareto dominating $\mathcal{T}$ which satisfies the property described above). So, for all agent h in $S \cap \tau_{\hat{h}}$, $S \cap \tau_{\hat{h}} \succ_h \tau_h$, which contradicts the fact that $\mathcal{T}$ was in $F(\succsim)$. ■

Proof of Theorem 3.2.

Let us note that $F(\succsim)$ is non-empty. In fact, the allocation $\tilde{\mathcal{T}}$ where $\tilde{\tau}_i = \{i\}$ for each i in N is always in $F(\succsim)$. Moreover, since $F(\succsim)$ is finite, there should be an allocation $\widehat{\mathcal{T}}$ which is Pareto efficient in $F(\succsim)$. By Lemma 4.2 $\widehat{\mathcal{T}}$ is Pareto efficient. So, $\widehat{\mathcal{T}}$ satisfies conditions (i) and (ii) in Proposition 4.1 and, hence, it is stable. ■

Coalition Formation and Essentiality

Before introducing a formal proof for Theorem 3.4, let us construct a function which will help us to understand how to find stable allocations in problems whose agents satisfy Essentiality. Let $E : 2^N \to 2^N$ be the function which associates with each set of individuals, $T \subseteq N$, the set of agents,

$$E(T) = \cup_{i \in T} \{S \subseteq N \mid S \text{ is essential for } i\}.$$

Note that, under Essentiality, a coalition structure $\mathcal{T}$ is individually rational if, and only if, (a) for each individual i, such that $\tau_i \neq \{i\}$, $E(\{i\}) \subseteq \tau_i$, and (b) there is no *self-sufficient* agent, say i, such that $\tau_i \neq \{i\}$. This statement gives us an idea of how to build individually rational allocations. In fact, imagine that there is no *self-sufficient* agent, then any coalition structure $\mathcal{T}$ such that $E(\tau^j) = \tau^j$, for each $\tau_j \in \mathcal{T}$ not being a singleton, satisfies individual rationality. Thus, loosely speaking, a fixed point for $E(\cdot)$ can be understood as a coalition that is not blocked by an individual. This property suggests the concept of *autonomous* coalitions.

Definition 4.3. *We say that a coalition T is autonomous for problem $\{N, \succsim\}$ if $E(T) = T$ and there is no coalition $T' \subset T$, $T' \neq \emptyset$, such that $E(T') = T'$.*

Note that any two different *autonomous* coalitions are disjoints, i.e., if S and T are two *autonomous* coalitions for $\{N, \succsim\}$, $S \neq T$, then $S \cap T = \emptyset$. Since the function $E(\cdot)$ has fixed points, in particular $E(N) = N$, the use of such a function and the idea of *autonomous* coalitions can be used to show the existence of stable structures for any problem satisfying Essentiality.

We are now ready to prove the statement of Theorem 3.4.

Proof of Theorem 3.4 Let $\{N, \succsim\}$ be a coalition formation problem whose agents' preferences satisfy Essentiality, and let $\mathcal{T}$ be the coalition structure which consists of the autonomous coalitions and the remaining agents as singletons. Suppose that $\mathcal{T}$ is not stable. Then there exists a blocking coalition T. If $|T| = 1$ then let $T = \{i\}$. Then i is in a non-singleton autonomous coalition, which is a contradiction. Thus, $|T| > 1$. If $T \supset S$, where S is an autonomous coalition, then for all $i \in S$, $S \succ_i T$, which is a contradiction. If $T \cap S \neq \emptyset$ for some autonomous coalition S, and $T \not\supset S$, where S is an autonomous coalition, then there exists $i \in T \cap S$ such that $\tau_i^e \setminus T \neq \emptyset$, which implies that $S \succ_i T$, a contradiction. Therefore, for all autonomous coalitions S, $T \cap S = \emptyset$. In this case, however, there exists $i \in T$ such that $\tau_i^e \setminus T \neq \emptyset$, which implies that $\{i\} \succ_i T$. Since this is a contradiction, $\mathcal{T}$ is stable. This proves the sufficiency of Essentiality.

In order to prove that the above stable coalition is unique, let $\mathcal{T}$ be a stable coalition structure. Let $S \subseteq N$ be an autonomous coalition. If S is a singleton then $S \in \mathcal{T}$, so assume that $|S| > 1$. Let $S' \subset N$ such that $S \not\subset S'$ and $S \cap S' \neq \emptyset$. Then there exists $i \in S \cap S'$ such that $\tau_i^e \setminus S' \neq \emptyset$. Thus, if $S' \in \mathcal{T}$ and $|S'| > 1$ then $\mathcal{T}$ is blocked by $\{i\}$. If $S' \in \mathcal{T}$ and $S' = \{i\}$ for some $i \in N$, then it follows that for all $j \in S$, $\{j\} \in \mathcal{T}$, since $\mathcal{T}$ is stable, and then $\mathcal{T}$ is blocked by S. Next, let $S' \subseteq N$ such that $S \subset S'$. Then, if $S' \in \mathcal{T}$ then $\mathcal{T}$ is blocked by S. This implies that $S \in \mathcal{T}$. Since this argument holds for all autonomous coalitions S, all autonomous coalitions are in $\mathcal{T}$.

Let N' denote the union of all autonomous coalitions. Let $i \in N$ such that $i \in N \setminus N'$. Then either (1) $\tau_i^e \cap N' \neq \emptyset$, or (2) there exists $j \in N \setminus N'$, $j \neq i$, such that $j \in \tau_i^e$, or both. If (1) does not hold then (2) holds, and the above possibilities are also true for all j's in (2). Given that there is a finite number of agents, we can repeat this argument until we find an agents $i' \in N$ for which (1) holds, since otherwise the set of agents that are examined contains an autonomous coalition, which is a contradiction. Then $\{i'\} \in \mathcal{T}$. Now let $N' := N' \cup \{i'\}$, and repeat the above argument, as many times as necessary, to show that all agents that are not in an autonomous coalition are in $\mathcal{T}$ as singletons. This prove that the unique stable coalition structure is the one consisting of the autonomous coalitions and the remaining singletons. ■