A discusión

BARGAINING, VOTING, AND VALUE*

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ABSTRACT

This paper addresses the following issue: If a set of agents bargain on a set of feasible alternatives 'in the shadow' of a voting rule, that is, any agreement can be enforced if a 'winning coalition' supports it, what general agreements are likely to arise? In other words: What influence can the voting rule used to settle (possibly non-unanimous) agreements have on the outcome of negotiations? To give an answer we model the situation as an extension of the Nash bargaining problem in which an arbitrary voting rule replaces unanimity to settle agreements by n players. This provides a setting in which a natural extension of Nash's solution is obtained axiomatically. Two extensions admitting randomization on voting rules based on two informational scenarios are considered.

Keywords: Bargaining, voting, value, bargaining in committees.

1 introduction

Nash (1950) proposes and characterizes axiomatically a cooperative 'solution' to the bargaining problem, in the spirit of von Neumann and Morgenstern's (1944) notion of 'value' of a zero-sum two-person game, as a rational expectation of the 'amount of satisfaction' or expected utility payo® of a 'highly rational' player engaging in a bargaining situation with another rational player. In Shapley (1953) a similar notion of value for transferable utility (TU) games is also proposed and characterized. Since then, several attempts to obtain a satisfactory 'solution' or a general notion of 'value' in the more general non-transferable utility (NTU) context have been made by di®erent authors. Harsanyi (1959, 1963), Shapley (1969), Kalai and Samet (1985), Maschler and Owen (1989, 1992) (see also Hart and Mas-Colell (1996)) have provided di®erent proposals[1]. As bargaining problems are two-person NTU games, and TU games are a particular case of the NTU model, the way of proceeding in all cases is to look for a notion of value for NTU games that coincides with Nash's solution and Shapley's value when restricted to bargaining problems and TU games, respectively. As is well known these extensions di®er, and there are no clear grounds for claiming superiority for any one of them over the others. Perhaps there is no de¯nite answer to this dispute. The reason for this might be an 'excess of abstraction' in the NTU model itself. The NTU model consists basically of a feasible set of utility vectors for each particular coalition. That is all. This seems to be enough in the simpler models that serve as term of reference: bargaining problems, where coalitions play no role, and TU games, where the feasible sets (or the con¯guration of players' preferences) are very particular. But in the general case the minimalistic NTU model seems too general to be sure ground for providing su±cient intuition.

The results presented in this paper appear to support this view. Nevertheless the original motivation of this work lies elsewhere. We are interested in understanding the in°uence that the voting rule used to settle agreements by a set of bargainers can have on the outcome of negotiations. More precisely (and this is exactly what this paper is about), if a set of agents bargain on a set of feasible alternatives 'in the shadow' of a voting rule, that is, any agreement can be enforced if a 'winning coalition' supports it, what general agreements are likely to arise? Or, put into Nash's classical terms, what agreements can a rational agent expect to arise when faced with the prospect of engaging in such a situation? The importance of the issue is clear in many contexts. It is often the case in a committee that uses a voting rule to make decisions that the ¯nal vote is merely the formal settlement of a bargaining process in which the issue to be voted upon has been adjusted to gain the acceptance of all members. It seems intuitively obvious that in such

[1] See McLean (2002) for a recent review.

cases the voting rule by means of which agreements are settled conditions the outcome of negotiations.

With this motivation in mind and in the spirit common to the classical results of von Neumann-Morgenstern, of Nash and of Shapley alluded to above, in this paper we explore an extension of Nash's (1950) model of a bargaining problem. We see Nash's original model as consisting of two ingredients, a set of (two) players with von Neumann-Morgenstern (1944) preferences over a set of feasible agreements, and a voting procedure (unanimity) to settle agreements. Thus the kind of situation we are interested in can be described by a natural generalization of this model (and its traditional extension to n players), by considering arbitrary voting rules[2] instead of unanimity. In this two-ingredient setting, on similar grounds to those in Nash (1950) or Shapley (1953), axioms such as e±ciency, anonymity, independence of irrelevant alternatives, invariance w.r.t. positive a±ne transformations and null-player can be adapted, keeping their meaning and motivation. Assuming these conditions entails a signi¯cant narrowing of the class of admissible answers or 'solutions', which is identi¯ed. It is shown that in fact all that remains is to settle the choice (within narrow limits) of an answer for the same issue in the particular case in which the con¯guration of preferences is TU-like. Then, in order to provide arguments in support of a choice, the class of problems under consideration is broadened by admitting random voting rules. The results in the deterministic case are consistently extended to this wider domain in two ways corresponding to two di®erent informational scenarios. It is then shown that the two extensions are incompatible, but there exists a unique solution in the deterministic domain that admits extensions in either sense which coincide for TU-bargaining problems.

The rest of the paper is organized as follows. In Section 2 the model is precisely formulated. In Section 3 the natural extension of Nash's and some of Shapley's axioms in the more general framework considered here are provided. Section 4 contains the ¯rst characterizing results, along with the relationships with Nash's solution and the Shapley value. The case in which random voting rules are admitted is considered in Section 5. Section 6 contains some remarks about the axioms, while the meaning of the results is discussed in Section 7 along with some lines of further research.

2 bargaining in the shadow of a voting rule

We are concerned with a situation in which a set of agents bargain on a set of feasible alternatives 'in the shadow' of a voting procedure. That is, any agreement can be enforced

[2] A setting including both a voting rule and a set of feasible agreements has already been considered by some authors but, as far as we know, with di®erent purposes and within a completely di®erent approach concerned with social choice issues (see Peleg (2002) for a recent overview).

if a 'winning coalition' supports it. The set $N = \{1, ..., n\}$ will label the seats of the decision procedure by means of which agreements are to be settled. As only yes/no voting is considered, a vote configuration can be represented by the set of 'yes'-voters. So, any $S \subseteq N$ represents the result of a vote in which players occupying seats in S voted 'yes' and those in $N \backslash S$ voted 'no'. An N-voting rule is specified by a set $W \subseteq 2^N$ of winning (i.e., which would lead to passing a decision) vote configurations such that (i) $N \in W$; (ii): $\emptyset \notin W$; (iii) If $S \in W$, then $T \in W$ for any T containing S; and (iv) If $S \in W$ then $N \backslash S \notin W$. $\mathcal{W}$ denotes the set of all such N-voting rules. For voting rule W, $M(W)$ denotes the set of minimal winning configurations, i.e., those that do not contain any other winning configuration. For any $S \in M(W)$ $(S \neq N)$, W_S^* denotes the voting rule that results from W by eliminating S from the set of winning configurations, that is, $W_S^* := W \backslash \{S\}$. For any permutation $\pi: N \to N$, πW denotes the voting rule $\pi W := \{\pi(S) : S \in W\}$. A voting rule W is symmetric if $\pi W = W$, for any permutation π.

If a set of n voters or players uses an N-voting rule they are labelled by the seats in N that they occupy, and we refer to the subset of players denoted by a subset $S \subseteq N$ as coalition S. Thus, depending on the context any $S \subseteq N$ will be referred to either as a vote configuration (of seats) or as a coalition (of players). We will also speak of winning coalitions for a given N-rule, with an obvious meaning. A seat/player $i \in N$ is said to be a null seat/player in W if, for any coalition S, $S \in W$ if and only if $S \backslash \{i\} \in W$.

We assume also that a set of n (N-labelled) players makes decisions by means of rule W in the following sense. They can reach any alternative within a set A, as well as any lottery over them, as long as: (i) a winning coalition supports it, and (ii) no player is imposed upon an agreement worse than the status quo, denoted a; where all players will remain if no winning coalition supports any agreement. It is also assumed that every player has expected utility (von Neumann and Morgenstern, 1944) (vNM) preferences over this set of lotteries, so that the situation can be summarized à la Nash in utility terms by a feasible set of utility vectors D, together with the particular vector d associated with the disagreement or status quo, as a summary of the situation concerning the players' decision. Accepting this simplification, the situation can be summarized by a pair (B, W), where $B = (D, d)$ is a classical n-person bargaining problem, and W is the N-voting rule to enforce agreements. In accordance with this interpretation we assume that $D_d := \{x \in D : x \geq d\}$[3] is bounded, and D is a closed, convex and comprehensive (i.e., $x \leq y \in D \Rightarrow x \in D$) set containing d, such that there exists some $x \in D$ s.t. $x > d$. $\mathcal{B}$ denotes the set of all such bargaining problems. For any permutation $\pi : N \to N$, $\pi B := (\pi(D), \pi(d))$ will denote the bargaining problem that results from B by π-permutation of its coordinates, so that for any $x \in \mathbb{R}^N$,

[3] We will write for any $x, y \in \mathbb{R}^N$, $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ for all $i = 1, ..., n$.

$\pi(x)$ denotes the vector in $\mathbb{R}^N$ s.t. $\pi(x)_{\pi(i)} = x_i$. A bargaining problem B is symmetric if $\pi B = B$, for any permutation π.

Thus, in this setting we are concerned with pairs $(B, W) \in \mathcal{B} \times \mathcal{W}$, each of which, consistently with the interpretation that accompanied its introduction, could properly be referred to as a bargaining problem B under rule W, or for short just a problem (B, W).

Before proceeding with the search for a 'value' or a 'solution' in this setting, let us first see how the model fits into the general NTU framework and how the classical bargaining problems and simple TU games fit into this model. Let $pr_S : \mathbb{R}^N \to \mathbb{R}^S$ denote the natural S-projection for $S \subseteq N$, and let $x^S := pr_S(x)$ for any $x \in \mathbb{R}^N$. On obvious grounds we can associate the NTU game $(N, V_{(B,W)})$ with every $(B, W) \in \mathcal{B} \times \mathcal{W}$, given by

$$V_{(B,W)}(S) := \begin{cases} pr_S(\{x \in D : x^{N\setminus S} = d^{N\setminus S}\}) & \text{if } S \in W, \\ pr_S(ch(d)) & \text{if } S \notin W, \end{cases}$$

where $B = (D, d)$, and $ch(d)$ denotes the comprehensive hull of $\{d\}$. For each $S \subseteq N$, $V_{(B,W)}(S)$ is the closed, convex and comprehensive subset of $\mathbb{R}^S$ containing all feasible utility vectors for coalition S. The n-person classical bargaining problem corresponds to the case in which W is the unanimity rule $W = \{N\}$, with N as the only winning coalition. While when the bargaining ingredient in the generalized model is the bargaining problem $\Lambda := (\Delta, 0)$, where $\Delta := \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq 1\right\}$, the associated NTU game $V_{(\Lambda,W)}$ is equivalent to a simple TU game. We will refer to $\Lambda = (\Delta, 0)$ as the (normalized) TU-bargaining problem. We reserve the notation Δ_N for the N-simplex $\Delta_N = \left\{x \in [0,1]^N : \sum_{i \in N} x_i = 1\right\}$. Note that in preferences terms the situation behind Λ is the following. There is no feasible agreement which Pareto-dominates any lottery over agreements b^i $(i = 1, 2, ..., n)$, where b^i denotes an agreement which is optimal for player i and indifferent to the status quo a for the other players (i.e., $u_i(b^i) = 1$, and $u_j(b^i) = u(a) = 0$ for $j \neq i$, for suitably chosen utilities). In this case $V_{(\Lambda,W)}$ is equivalent to the simple TU game representing the rule v_W, given by

$$v_W(S) := \begin{cases} 1 & \text{if } S \in W, \\ 0 & \text{if } S \notin W. \end{cases}$$

Thus the family of associated NTU games $\{(N, V_{(B,W)}) : (B, W) \in \mathcal{B} \times \mathcal{W}\}$ properly contains all classical bargaining problems and all simple superadditive games[4]. Nevertheless, in spite of the possibility of embedding the family of problems under consideration into a subclass of NTU games, we will deal with the model in terms of its constituent elements, B and W.

[4] A TU game $v : 2^N \to \mathbb{R}$ is superadditive if $v(S \cup T) \geq v(S) + v(T)$, for all disjoint $S, T \subseteq N$.

3 rationality conditions for a 'value' or a 'solution'

In the class of situations described by this model the question addressed by Nash (1950) and Shapley (1953) in their respective cases can also be addressed: What is the 'amount of satisfaction' or utility that a rational player can expect from such a situation, or, in classical terms, what is the value for any player of the prospect of engaging in a situation such as this? We will proceed as in the two seminal papers by stating reasonable conditions that will narrow the possible options. Thus, we will impose some conditions on a map $\Phi : \mathcal{B} \times \mathcal{W} \to R^N$, for vector $\Phi(B, W) \in R^N$ to be considerable as a rational agreement, or better as a reasonable expectation of utility levels of a general agreement in a bargaining situation B under voting rule W. To begin with, in view of the interpretation in terms of the underlying situation stated in section 2, we build the requirements of being feasible and no worse than the status quo for any player into the very notion of solution, that is, we impose as prerequisites: $\Phi(B, W) \in D$ (feasibility), and $\Phi(B, W) \geq d$ (individual rationality); if $B = (D, d)$. In addition to this we require the following conditions, all but one of which are direct adaptations of Nash's and Shapley's characterizing properties:

1. Efficiency (Ef): For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, there is no $x \in D$, s.t. $x > \Phi(B, W)$.

2. Anonymity (An): For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and any permutation $\pi : N \to N$, and any $i \in N$, $\Phi_{\pi(i)}(\pi(B, W)) = \Phi_i(B, W)$; where $\pi(B, W) := (\pi B, \pi W)$.

3. Independence of irrelevant alternatives (IIA): Let $B, B' \in \mathcal{B}$, with $B = (D, d)$ and $B' = (D', d')$, such that $d' = d$; $D' \subseteq D$ and $\Phi(B, W) \in D'$, then $\Phi(B', W) = \Phi(B, W)$, for any $W \in \mathcal{W}$.

4. Invariance w.r.t. positive affine transformations (IAT): For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, and all $\alpha \in R^N_{++}$ and $\beta \in R^N$,

$$\Phi(\alpha * B + \beta, W) = \alpha * \Phi(B, W) + \beta,$$

where $\alpha * B + \beta = (\alpha * D + \beta, \alpha * d + \beta)$; denoting $\alpha * x := (\alpha_1 x_1, ..., \alpha_n x_n)$; and $\alpha * D + \beta := \{\alpha * x + \beta : x \in D\}$.

5. Null player (NP): For all $(B, W) \in \mathcal{B} \times \mathcal{W}$, if $i \in N$ is a null player in W, then $\Phi_i(B, W) = d_i$.

The readers can see for themselves the precise correspondence of axioms 1 to 5 with some of Nash's and Shapley's axioms. But it is worth noting some subtle differences arising in this setup. Observe that Ef, IIA and IAT (adaptations of Nash's axioms) concern the feasible set, while An (adapted from Nash's and Shapley's anonymity[5]) and

[5] In Nash (1953) anonymity replaces symmetry. We prefer this simpler condition to the slightly weaker symmetry in Nash (1950), which can also be adapted to this setting

NP (from Shapley's system) concern both. We omit the arguments in support of each of these conditions, which can be found in Nash's and Shapley's papers[6]. It may be worth remarking though that An entails a consistent relabelling of voters in B and seats in W. Note also that it seems natural in our setting to set null players' expectations to zero, or more precisely to the status quo level, given their null capacity to influence the outcome given the voting rule according to which final agreements are enforced.

4 first characterization

In this section we show how assuming the above five conditions drastically restrict the class of admissible 'solutions' or 'values' $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$.

Denote by $Nash(B)$ the Nash (1950) bargaining solution of an n-person bargaining problem $B = (D; d)$, that is,

$$Nash(B) = \arg\max_{x \in D_d} \prod_{i \in N} (x_i - d_i);$$

and by $Nash^w(B)$ the w-weighted asymmetric Nash bargaining solution (Kalai, 1977) of the same problem for a vector of nonnegative weights $w = (w_i)_{i \in N}$,

$$Nash^w(B) = \arg\max_{x \in D_d} \prod_{i \in N} (x_i - d_i)^{w_i}.$$

In fact, if any of the weights is zero, the w-weighted Nash solution may not be unique under the conditions assumed in the bargaining problems. This difficulty can be overcome in two ways: Either by assuming a condition of 'non-levelness' on the feasible set, or by imposing the disagreement 'payoff' for players whose weight is zero. That is, redefining

$$Nash_i^w(B) := \begin{cases} \arg_i \max_{x \in D_d} \prod_{j \in J} (x_j - d_j)^{w_j} & \text{if } i \in J; \\ d_i & \text{if } i \in N \setminus J, \end{cases} \tag{1}$$

where $J = \{i \in N : w_i > 0\}$. In what follows we refer always to definition (1).

Proposition 1 Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be a solution/value that satisfies Eff, An, IIA and IAT, then there exists an anonymous (i.e., such that $\varphi_{\pi(i)}(\pi W) = \varphi_i(W)$, for any permutation π) map $\varphi : \mathcal{W} \to \Delta_N$ such that, for all $(B; W) \in \mathcal{B} \times \mathcal{W}$,

$$\Phi(B; W) = Nash^{\varphi(W)}(B). \tag{2}$$

In particular, if W is a symmetric voting rule, $\Phi(B; W) = Nash(B)$.

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[6] For a careful discussion of Nash's axioms see, e.g., chapter 1 in Binmore (1998).

Proof. Fix a rule W. Then E®, IIA and IAT become for ©(¢; W) : B ! R^N exactly Nash's (1950) corresponding characterizing conditions of his solution in the domain B of classical bargaining problems. In view of Kalai's (1977) characterization it must be ©(¢; W) = $Nash^w$(¢); for some vector of nonnegative weights, which, to avoid indeterminacy, we can assume to be 'normalized', that is, w 2 $¢_N$. In general w is dependent on W, so that we can write w = ' (W), for some map ' : W ! $¢_N$. In particular if B = ¤, as ¼¤ = ¤, An imposes anonymity on ', i.e., $'_{¼(i)}$(¼W) = $'_i$(W); for any permutation ¼. Moreover, if W is a symmetric rule, An becomes exactly Nash's anonymity for ©(¢; W), so that in this case, for all B 2 B, ©(B; W) = Nash(B): ■

Thus E®, An, IIA and IAT, narrow the class of possible answers to solutions of the form (2), where ' : W ! $¢_N$ is an anonymous function of the voting rule. We leave to the reader the simple proof of the following lemma, which will be of use:

Lemma 1 For any vector of nonnegative weights w 2 R^N (w 6= 0), $Nash^w$(¤) = w̄; where w̄ := w= $\sum_{i2N} w_i$ denotes the normalization of w.

As a corollary we have the following

Proposition 2 Let © : B £ W ! R^N be a solution/value that satis¯es E®, An, IIA, and IAT, then

$$©(B; W) = Nash^{©(¤;W)}(B): \quad (3)$$

Proof. By Proposition 1, there must exist an anonymous map ' : W ! $¢_N$ such that ©(B; W) = $Nash^{'(W)}$(B): In particular, for B = ¤, in view of Lemma 1, it must be

$$©(¤; W) = Nash^{'(W)}(¤) = '(W);$$

for all W 2 W. Note that ©(¤; W) 2 $¢_N$ by E®. ■

In other words: Assuming E®, An, IIA, and IAT, the solution, given by (3), will be unique as soon as ©(¤; ¢) is speci¯ed. As stated in Section 2, in the case of the TU-bargaining problem ¤, for any voting rule W, the associated NTU game $V_{(¤;W)}$ is equivalent to the simple TU game v_W s.t. v_W(S) = 1 if and only if S 2 W. Thus, the solution is so far determined up to the choice of a value on the domain of simple superadditive games. But Propositions 1 and 2 impose certain conditions on this value. Namely, 'e±-ciency' and 'anonymity'. Finally, the null player (NP) condition on ©, imposes the null player condition on this value. Thus, the following theorem summarizes the situation.

Theorem 1 *Let $\Phi : \mathcal{B} \times \mathcal{W} \rightarrow \mathbb{R}^N$ be a solution/value that satisfies efficiency (Eff), anonymity (An), independence of irrelevant alternatives (IIA), invariance w.r.t. affine transformations (IAT) and null player (NP), then*

$$\Phi(B, W) = Nash^{\Phi(*, W)}(B),$$

where $\Phi(*, \cdot) : \mathcal{W} \rightarrow \mathbb{R}^N$ satisfies efficiency, anonymity and null player.

Therefore, any map $\Phi(*, \cdot) : \mathcal{W} \rightarrow \mathbb{R}^N$ that satisfies efficiency, anonymity and null player would fit into formula (3) and yield a solution $\Phi(B, W)$ that satisfies the five rationality conditions. The conditions on $\Phi(*, \cdot)$ bring immediately to mind the Shapley (1953) value, or more specifically in the context of simple games, the Shapley-Shubik (1954) index[7]. Dubey (1975) characterized the Shapley value on the domain of simple games (i.e., the Shapley-Shubik index) using the lattice property of 'transfer', which in terms of voting rules can be stated as follows (Laruelle and Valenciano, 2001):

6. Transfer (T): For any two rules $W, W' \in \mathcal{W}$, and all $S \in M(W) \cap M(W')$ $(S \neq N)$:

$$\Phi(*, W) - \Phi(*, W_S^*) = \Phi(*, W') - \Phi(*, W_S'^*). \tag{4}$$

Denote by $Sh(v)$ the Shapley (1953) value of a TU game v, given by

$$Sh_i(v) = \sum_{S : S \subseteq N} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \backslash i)),$$

and by $Sh(W)$ the Shapley-Shubik (1954) index of a voting rule W, i.e., the Shapley value of the associated simple game v_W. We have the following result.

Proposition 3 *Let $\Phi : \mathcal{B} \times \mathcal{W} \rightarrow \mathbb{R}^N$ be a solution/value that satisfies Eff, An, NP and T, then for any voting rule $W \in \mathcal{W}$, $\Phi(*, W) = Sh(W)$.*

Proof. For any $W \in \mathcal{W}$, as stated in Section 2, the associated NTU game $V_{(*, W)}$ is equivalent to the simple TU game v_W s.t. $v_W(S) = 1$ if and only if $S \in W$. And, as is well known, efficiency, anonymity, null player and transfer characterize the Shapley (-Shubik) value in the domain of simple (superadditive or not) games (Dubey (1975), see also Laruelle and Valenciano (2001)). It can then be easily checked that in our setting conditions Eff, An (recall that $*$ is symmetric), NP and T become their homonymous for $\Phi(*, \cdot) : \mathcal{W} \rightarrow \mathbb{R}^N$. Thus $\Phi(*, W) = Sh(v_W) = Sh(W)$ for any voting rule. ■

Then as a corollary of Theorem 1 and Proposition 3 we have:

[7] But there are other alternatives, for instance, the normalization of any semivalue (Dubey, Neyman and Weber, 1981) meets these conditions. Also some 'power indices,' as the Holler-Packel (1983) index.

Theorem 2 There exists a unique solution/value $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ that satisfies efficiency (Eff), anonymity (An), independence of irrelevant alternatives (IIA), invariance w.r.t. affine transformations (IAT), null player (NP) and transfer (T), and it is given by

$$\Phi(B, W) = Nash^{Sh(W)}(B). \tag{5}$$

Proof. Existence: For any $(B, W) \in \mathcal{B} \times \mathcal{W}$, $Nash^{Sh(W)}(B)$ exists by the compactness of D_d, whose convexity makes it unique under definition (1). It is easy to see then that the solution $\Phi(B, W) := Nash^{Sh(W)}(B)$ satisfies Eff, An, IIA, IAT and NP. As to T, let $W, W' \in \mathcal{W}$, and $S \in M(W) \cap M(W')$ $(S \neq N)$. In view of Lemma 1 and the fact that the Shapley value satisfies transfer, we have

$$\Phi(\ast, W) - \Phi(\ast, W_S^\ast) = Nash^{Sh(W)}(\ast) - Nash^{Sh(W_S^\ast)}(\ast) = Sh(W) - Sh(W_S^\ast)$$

$$= Sh(W') - Sh(W_S'^\ast) = Nash^{Sh(W')}(\ast) - Nash^{Sh(W_S'^\ast)}(\ast) = \Phi(\ast, W') - \Phi(\ast, W_S'^\ast).$$

Uniqueness[8]: Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be a value or solution that satisfies Eff, An, IIA, IAT, NP and T. By Theorem 1 it must be $\Phi(B, W) = Nash^{\Phi(\ast, W)}(B)$. And by Proposition 3, $\Phi(\ast, W) = Sh(W)$. ■

Note that NP and T become empty requirements when W is any fixed symmetric rule, while conditions IIA and IAT become empty requirements when fixing $B = \ast$. Then the second part of Proposition 1 can be rephrased like this: the characterizing axioms in Theorem 2 when restricted to $\Phi(\cdot, W) : \mathcal{B} \to \mathbb{R}^N$ for any fixed symmetric rule, become Nash's axiomatic system. Proposition 3 can also be rephrased like this: the characterizing axioms in Theorem 2 when restricted to $\Phi(\ast, \cdot) : \mathcal{W} \to \mathbb{R}^N$ become Shapley-Dubey's characterizing system of the Shapley value (or Shapley-Shubik index) in $\mathcal{W}$. In other words, Theorem 2 integrates Nash's and Shapley-Dubey's characterizations into one. But it goes further beyond both characterizations, yielding a surprising solution to the more complex problem under consideration given by (5).

Nevertheless, there is still the question of the compellingness of transfer (T), which is far from obvious. In words, this condition postulates that the effect of eliminating a minimal winning coalition from the set of winning coalitions is the same whatever the voting rule (one of whose minimal winning coalitions is that coalition) as long as the bargaining component is the normalized TU-bargaining problem. Why this should be so? To address this point we introduce risk in the voting rule.

[8]The proof of uniqueness relies upon some sort of 'folk theorems'. For instance, Kalai's (1977) result, alluded to in the proof of Proposition 1, was in fact proved in his paper only for two-person bargaining problems and positive weights. Therefore we consider it convenient to give (in the Appendix) an independent proof to dissipate any doubts and make the paper self-contained.

5 rationality under risk in the voting rule

Theorem 1 narrows the class of admissible solutions, i.e., those satisfying the five rationality conditions introduced in Section 2, to the family given by (3), where $\Phi(*;\cdot) : \mathcal{W} \to \mathbb{R}^N$ verifies efficiency, anonymity and null player. In order to go one step further within this family we need either to justify the transfer condition or to introduce additional compelling conditions. With this purpose we will enrich the model by also admitting random voting rules. There are two points of view to motivate this extension of the domain. On the one hand, it is often the case that a committee uses different voting rules to decide upon different classes of issue, and there may exist uncertainty about which rule should be used to decide upon an issue. On the other hand, at the very foundations of Nash's model is the consideration of risk in the feasible agreements, inherent to the description of vNM-rational players (though in practice negotiators tend to avoid random agreements), which constrains the admissible utility functions. In other words, even if it is assumed that for any point in the feasible set there exists a deterministic agreement yielding that utility vector, the model only makes full sense if these are assumed to be VNM utilities. The situation under consideration here is an extension of the one modeled by Nash in which the new element is the voting rule. Introducing risk in the voting rule will allow us similarly (even if we were only interested in the case of deterministic voting rules) to further constrain the degrees of freedom in the choice of the 'value' function.

Thus, the second element in the model will now be in general a lottery over the set of all N-voting rules, denoted by $L(\mathcal{W})$. In this wider setting (we identify each deterministic voting rule with the corresponding degenerated lottery that assigns probability 1 to that rule) a problem will consist of a pair $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$. Any lottery $\lambda \in L(\mathcal{W})$ will be represented by a map $\lambda : \mathcal{W} \to [0;1]$ s.t. $\sum_{W \in \mathcal{W}} \lambda(W) = 1$; where $\lambda(W)$ represents the probability of voting rule W. Note that any given $\lambda \in L(\mathcal{W})$ induces a probability of each coalition being winning. We will also use the following notation: For any $S \subseteq N$,

$$\lambda_S := \sum_{W : S \in W \in \mathcal{W}} \lambda(W);$$

that is, λ_S is the probability of S being winning for the random procedure λ. The distinction between a voting rule W and its associated simple game v_W, allows us to avoid ambiguity: $\mu W + (1-\mu)W'$ represents a random voting rule, while $\mu v_W + (1-\mu)v_{W'}$ is a TU game.

As in the case of deterministic voting rules, for any $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$, we can associate with it the NTU game $(N; V_{(B;\lambda)})$, such that for any $S \subseteq N$,

$$V_{(B;\lambda)}(S) := \begin{cases} \lambda_S \, pr_S(\{x \in D : x^{N \setminus S} = d^{N \setminus S}\}) + (1-\lambda_S) d^S & \text{if } \lambda_S \neq 0; \\ pr_S(ch(d)) & \text{if } \lambda_S = 0. \end{cases}$$

This is consistent with the meaning of D: if λ_S is the probability of S being winning, this coalition can guarantee expected utilities for its members within the specified set. While for the TU-bargaining case $(*;\lambda)$, the associated NTU game $V_{(*;\lambda)}$ is equivalent to the TU game

$$v_\lambda(S) := \lambda_S; \text{ for all } S \subseteq N.$$

Note also that

$$v_\lambda = \sum_{W \in \mathcal{W}} \lambda(W)\, v_W. \qquad (6)$$

But bear in mind that not all n-person TU games are generated in this way. In view of (6), only those in the convex-hull (in the space $\mathbb{R}^{2^n - 1}$) of the set of simple superadditive games are generated as particular cases of $V_{(*;\lambda)}$. And more importantly, unlike the deterministic case, the associated NTU game $(N; V_{(B;\lambda)})$ encapsulates in general less information than the pair $(B;\lambda)$; because different lotteries in $L(\mathcal{W})$ can yield the same $v_\lambda = (\lambda_S)_{S \subseteq N}$.

We now address the issue of a solution/value $\Phi : \mathcal{B} \times L(\mathcal{W}) \to \mathbb{R}^N$ in this wider setting. We consider two scenarios for this extension supported on two alternative assumptions about the information of the players on the bargaining environment:

Scenario 1: The configuration of preferences B is common knowledge, but only the probabilities of each coalition being winning (i.e., $(\lambda_S)_{S \subseteq N}$) are common knowledge.

Scenario 2: Both elements of the bargaining environment (i.e., the pair $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$) are common knowledge.

In Scenario 1 players know the probabilities of different coalitions being winning but are unaware of the actual lottery. Therefore their expectations have to be founded on the $(2^n - 1)$-vector of probabilities $v_\lambda = (\lambda_S)_{S \subseteq N}$, instead of on the random voting rule itself. Thus in this scenario it is only consistent requiring:

7. Coalitional expectations dependence (CED): For all $\lambda, \lambda' \in L(\mathcal{W})$, such that for any $S \subseteq N$; $\lambda_S = \lambda'_S$, $\Phi(B;\lambda) = \Phi(B;\lambda')$ for all B.

As the reader can easily check, there is no difficulty in extending Eff, An, IIA, IAT and NP to the wider domain $\mathcal{B} \times L(\mathcal{W})$, where they keep their meaning and motivation[9]. Combining this extension with CED we obtain the following extension of Theorem 1, whose proof, entirely similar, we omit

Theorem 3 *Let $\Phi : \mathcal{B} \times L(\mathcal{W}) \to \mathbb{R}^N$ be a solution/value that satisfies Eff, An, IIA, IAT, NP and CED on $\mathcal{B} \times L(\mathcal{W})$, then, for all $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$,*

$$\Phi(B;\lambda) = Nash^{\Phi(*;\lambda)}(B);$$

[9] For instance, now anonymity involves a consistent relabelling of B and $v_\lambda = (\lambda_S)_{S \subseteq N}$; and a null player is a player whose entering or leaving any coalition never modifies its probability of being winning.

where $\Phi(\cdot;\cdot) : L(\mathcal{W}) \to \mathbb{R}^N$ satisfies efficiency, anonymity, null player and CED. In particular, for all $W \in \mathcal{W}$, $\Phi(B;W) = Nash^{\Phi(\cdot;W)}(B)$.

Now consider Scenario 2, where both B and λ are common knowledge. In this case it seems reasonable to assume that when rational players are faced with the prospect of using a random voting rule λ they should, whatever the B, reach an agreement before applying the random rule that is better than the payoff expected if they postpone the agreement to carry out the lottery, provided that they have the chance to do so. We have then the following condition of rationality under random voting rules: For all $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$,

$$\Phi(B;\lambda) \geq \sum_{W \in \mathcal{W}} \lambda(W)\Phi(B;W).$$

If so, this expected payoff can be seen as a reserve option. But this is as if assuming that players take $d_\lambda := \sum_{W \in \mathcal{W}} \lambda(W)\Phi(B;W)$, as the actual disagreement point, that is, $B_\lambda := (D;d_\lambda)$ as the actual problem. In this case it seems reasonable to use 'by default' the unanimity rule ($\mathcal{W} = \{N\}$) for bargaining the 'extra' utilities. So one step further is requiring

8. Strong rationality under random voting rules (SRR): For all $(B;\lambda) \in \mathcal{B} \times L(\mathcal{W})$,

$$\Phi(B;\lambda) = \Phi(B_\lambda;\{N\}).$$

Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be any solution in the family characterized in Theorem 1. Its unique extension to $\mathcal{B} \times L(\mathcal{W})$ satisfying condition SRR is given by

$$\Phi(B;\lambda) = \Phi(B_\lambda;\{N\}) = Nash^{\Phi(\cdot;\{N\})}(B_\lambda) = Nash(B_\lambda).$$

And note that SRR (along with Eff on $\mathcal{B} \times \mathcal{W}$) implies the following condition on $\Phi(\cdot;\cdot)$:

8'. Weak rationality under random voting rules (WRR): For all $\lambda \in L(\mathcal{W})$,

$$\Phi(\cdot;\lambda) = \sum_{W \in \mathcal{W}} \lambda(W)\Phi(\cdot;W).$$

Thus we have an alternative extension of Theorem 1:

Theorem 4 Let $\Phi : \mathcal{B} \times L(\mathcal{W}) \to \mathbb{R}^N$ be a solution that satisfies Eff, An, IIA, IAT and NP on $\mathcal{B} \times \mathcal{W}$, and SRR on $\mathcal{B} \times L(\mathcal{W})$, then for all $B \in \mathcal{B}$, and all $\lambda \in L(\mathcal{W})$,

$$\Phi(B;\lambda) = Nash(B_\lambda);$$

where $B_\lambda = (D;d_\lambda)$, with $d_\lambda = \sum_{W \in \mathcal{W}} \lambda(W)\Phi(B;W)$; and $\Phi(B;W) = Nash^{\Phi(\cdot;W)}(B)$, where $\Phi(\cdot;\cdot) : L(\mathcal{W}) \to \mathbb{R}^N$ satisfies efficiency, anonymity, null player and WRR.

Thus, Theorems 3 and 4 provide two alternative extensions of Theorem 1, and of the family characterized by it. But in view of the different informational scenarios that support conditions SRR and CED it is not surprising that, as we will show, these extensions are incompatible in the following sense: The intersection of the families characterized by either theorem is empty. But surprisingly enough there exists a unique solution in the deterministic domain (i.e., $B \times W$) that admits extensions in either sense which coincide for TU-bargaining problems. Thus we have a two-faced result, existence and uniqueness on one side and impossibility on the other.

Let us see first that WRR and CED imply transfer (T) on the subdomain $\{\delta\} \times W$.

Lemma 2 Let $\Phi : B \times L(W) \to \mathbb{R}^N$ be a solution/value such that $\Phi(\delta; \cdot) : L(W) \to \mathbb{R}^N$ satisfies WRR and CED, then $\Phi(\delta; \cdot)$ satisfies T.

Proof. Let $W, W' \in W$, and $S \in M(W) \cap M(W')$ $(S \neq N)$. It can immediately be checked that the lotteries $\frac{1}{2}W + \frac{1}{2}W_S'^{*}$ and $\frac{1}{2}W_S^{*} + \frac{1}{2}W'$ assign the same probability of being winning to every coalition. Then, combining CED and WRR, we have

$$\frac{1}{2}\Phi(\delta; W) + \frac{1}{2}\Phi(\delta; W_S'^{*}) = \Phi(\delta; \frac{1}{2}W + \frac{1}{2}W_S'^{*})$$
$$= \Phi(\delta; \frac{1}{2}W_S^{*} + \frac{1}{2}W') = \frac{1}{2}\Phi(\delta; W_S^{*}) + \frac{1}{2}\Phi(\delta; W'):$$

Which yields transfer (4) for $\Phi(\delta; \cdot)$. ■

Then we have the following existence and uniqueness result:

Theorem 5 There exists a unique solution $\Phi : B \times W \to \mathbb{R}^N$ which admits extensions to $B \times L(W)$ satisfying the conditions of Theorem 3 and Theorem 4 which coincide for TU-bargaining problems, and it is given by $\Phi(B; W) = Nash^{Sh(W)}(B)$:

Proof. Let $\Phi : B \times W \to \mathbb{R}^N$ be the restriction of two solutions Φ_1 and Φ_2 on $B \times L(W)$ belonging to the families characterized in Theorems 3 and 4 respectively, and such that $\Phi_1(\delta; \lambda) = \Phi_2(\delta; \lambda)$, for all $\lambda \in L(W)$. Then Φ_1 and Φ_2 must satisfy CED and WRR on $\{\delta\} \times L(W)$, where they coincide. Then, by Lemma 2, they and consequently Φ satisfy T, so that in view of Theorem 2 it must be $\Phi(B; W) = Nash^{Sh(W)}(B)$: On the other hand, it is easy to see that this solution can be extended consistently and uniquely to two maps Φ_1 and Φ_2 on $B \times L(W)$ under the conditions of either Theorem 3 and Theorem 4 respectively. For these extensions we have, for all λ;

$$\Phi_2(\delta; \lambda) = Nash(\delta_\lambda) = Nash(\cdot; \sum_{W \in W} \lambda(W) Sh(W))$$

$$= \sum_{W \in \mathcal{W}} \lambda(W) Sh(W) = Sh(v_\lambda) = Nash^{Sh(v_\lambda)}(¤) = \Phi_1(¤; \lambda). \blacksquare$$

As a corollary we have:

Theorem 6 *There is no solution $\Phi : \mathcal{B} \times \mathcal{L}(\mathcal{W}) \to \mathbb{R}^N$ that satisfies the conditions of Theorem 3 and those of Theorem 4. In other words, the intersection of the families characterized in either theorem is empty.*

Proof. Assume $\Phi : \mathcal{B} \times \mathcal{L}(\mathcal{W}) \to \mathbb{R}^N$ satisfies the conditions of Theorems 3 and 4. In view of Theorem 5, $\Phi(B; W) = Nash^{Sh(W)}(B)$. And it is easy to see that the only extension consistent with Theorem 3 is $\Phi(B; \lambda) = Nash^{Sh(v_\lambda)}(B)$. But this solution does not satisfy SRR. To see this it suffices to provide an example $(B; \lambda)$ for which

$$Nash^{Sh(v_\lambda)}(B) \neq \sum_{W \in \mathcal{W}} \lambda(W) Nash^{Sh(W)}(B). \qquad (7)$$

The following example serves this purpose[10]: Let $N = \{1, 2, 3\}$, and $B = (D, 0)$ the 3-person bargaining problem in which D is the comprehensive hull of the convex hull of the set:

$$\left\{ (1, 0, 0), (0, 1, 0), (\frac{2}{3}, 0, \frac{1}{3}), (0, \frac{2}{3}, \frac{1}{3}) \right\}.$$

And let λ be the random voting rule that assigns probability $1/3$ to each of the three unanimity rules W^{12}, W^{13}, and W^{23}, where W^{ij} denotes the rule whose only minimal winning coalition is $\{i, j\}$. Then, as $Sh(v_\lambda) = (1/3, 1/3, 1/3)$, we have

$$Nash^{Sh(v_\lambda)}(B) = Nash(B) = (1/3, 1/3, 1/3),$$

while

$$\sum_{W \in \mathcal{W}} \lambda(W) Nash^{Sh(W)}(B) = \frac{1}{3}(\frac{2}{3}, 0, \frac{1}{3}) + \frac{1}{3}(0, \frac{2}{3}, \frac{1}{3}) + \frac{1}{3}(\frac{1}{2}, \frac{1}{2}, 0) = (\frac{3.5}{9}, \frac{3.5}{9}, \frac{2}{9}).$$

Thus we have (7). $\blacksquare$

6 some remarks about axioms and results

1. The following solutions show the independence of the axioms used in Theorem 2 as well as WRR and CED.

$\Phi(B; \lambda) = d$, satisfies all but Eff.

$\Phi(B; \lambda) = Nash^{Sh^\omega(v_\lambda)}(B)$, where Sh^ω is any nonsymmetric ω-weighted Shapley value (Kalai and Samet, 1987), satisfies all but An.

[10] We thank Abraham Neyman, who provided a first example for (7).

$\Phi(B;\lambda) = Nash(B)$, satisfies all but NP.

$\Phi(B;\lambda) = d + \hat{t}\, Sh(v_\lambda)$, where $\hat{t} = \max\{t \in \mathbb{R} : d + t\, Sh(v_\lambda) \in D\}$; satisfies all but IAT.

$\Phi(B;\lambda) = d + \hat{t}\,((m - d) * Sh(v_\lambda))$, where $\hat{t} = \max\{t : d + t\,((m - d) * Sh(v_\lambda)) \in D\}$ with $m \in \mathbb{R}^N$ s.t. $m_i = \max pr_i(D_d)$, satisfies all but IIA (note this family yields Kalai-Smorodinsky (1975) solution as the particular case $\Phi(\cdot;W)$ when W is symmetric).

$\Phi(B;\lambda) = Nash^{Bz(v_\lambda)}(B)$, where Bz is the Banzhaf (1965) index (or any semivalue (Dubey, Neyman and Weber, 1981) different from Sh), satisfies all but WRR (and T).

$\Phi(B;\lambda) = Nash^{HP(\lambda)}(B)$, where HP is the extension of Holler-Packel's (1983) index given by $HP(\lambda) := \sum_{W \in \mathcal{W}} \lambda(W) HP(W)$, where $HP_i(W) = \frac{w_i}{\sum_i w_i}$ and w_i is the number of minimal winning coalitions in W containing i, satisfies all but CED (and T).

2. In Theorem 2 transfer (T) can be replaced by the weaker (in the presence of anonymity) condition introduced in Laruelle and Valenciano (2001) of symmetric-gain loss, which in the present setting can be restated like this:

(SymGL) For any voting rule $W \in \mathcal{W}$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(*;W) - \Phi_i(*;W^*_S) = \Phi_j(*;W) - \Phi_j(*;W^*_S)$$

for any two voters $i, j \in S$; and any two voters $i, j \in N \setminus S$.

That is, the effect of eliminating a (minimal) winning configuration from the list that specifies the voting rule is equal on any two voters belonging (not belonging) to it. It is easy to check that this condition can replace T in Proposition 3, and in Theorem 2. If the weakness of the axioms were our main goal, this would be a better choice. But, although SymGL also beats transfer in simplicity (it involves the modification of a single voting rule), it entails equating gains (or losses) of utility of different agents. Nevertheless the fairness flavor of this condition may be interesting for a normative interpretation of the solution characterized in Theorem 2.

3. In Theorem 4, though technically unnecessary, the axioms Eff, An, IAT and NP can be extended to and required on the wider domain $B \times L(\mathcal{W})$ (instead of requiring them only on $B \times \mathcal{W}$). But the natural extension of IIA to $B \times L(\mathcal{W})$ is inconsistent with SRR. This can be easily explained: Requiring SRR implies that some alternatives apart from $\Phi(B;\lambda)$ are no longer 'irrelevant'. Namely, by SRR, in the determination of $\Phi(B;\lambda)$ the points in $\{\Phi(B;W) : \lambda(W) \neq 0\}$ are obviously relevant, and these points change with the feasible set.

4. The impossibility result of Theorem 6 is easy to understand: The logical clash between SRR and CED is not surprising given the inconsistency of the informational environments that support them: common knowledge either of λ (SRR) or of v_λ (CED).

This makes the compatibility of these conditions on $\{*\} \times L(W)$ as established by Theorem 5 even more surprising .

7 conclusion

Theorem 1 extends the Nash bargaining solution and provides an endogenous justi¯cation of the 'weights' in Kalai's (1977) asymmetric Nash solutions in the complete information context[11]. Kalai's solutions emerge when symmetry is dropped in Nash's system, or still assuming symmetry in an adequately 'replicated' problem (Kalai, 1977), as if each player negotiated on behalf of a number of players, as is usually the underlying situation when a nonsymmetric rule is used to make decisions. Binmore (1998, p. 78) justi¯es the asymmetric Nash solutions as re°ecting the di®erent 'bargaining power' of the players 'determined by the strategic advantages conferred on players by the circumstances under which they bargain', and uses the term 'bargaining power' to refer to the players' weights[12]. In our case these circumstances consist of the voting rule that governs negotiations. Thus the extension provided by Theorem 1 re-opens the old power indices' 'contest' in a richer and more interesting setting than the traditional one of bare simple games. Now they (i.e., e±cient, anonymous values for simple games satisfying the null player condition) compete to adequately represent the 'bargaining power' (in a precise and relevant sense) that a voting rule confers to its users in a bargaining committee.

Theorem 2 singles out the Shapley-Shubik index by integrating two axiomatic systems into a single, consistent one, which when restricted to classical bargaining problems becomes Nash's system, and when restricted to simple games becomes a characterizing system (Dubey-Shapley's) of the Shapley value. As a result the solution emerging turns out to be consistent with both Nash's solution and Shapley's value. But in general no solution in the NTU literature coincides with the one obtained here on axiomatic grounds in the class of NTU games associated with the class of two-ingredient models considered by us[13]. This seems to corroborate the impression of the excessive abstraction and conse-

[11] See Harsanyi and Selten (1972) for the incomplete information case.

[12] This interpretation is consistent with the outcome of Rubinstein's (1982) alternating o®ers model within the noncooperative approach (see Binmore, 1987). It is also consistent with the interpretation proposed by Valenciano and Zarzuelo (1994) within Rubinstein, Safra and Thomson's (1992) preference-based model.

[13] The solution obtained depends on the bargaining problem and the players' weights (given by the Shapley-Shubik index of the voting rule). Consequently, the bargaining problem can be modi¯ed in many ways so that the feasible set for some coalition(s) changes without the solution changing, unlike other solutions of the associated NTU game. Nevertheless, as Sergiu Hart pointed out, the solution characterized here can be seen as the Shapley NTU value-like of an NTU game by associating the whole feasible set with

quent lack of intuitive basis of the bare NTU model: unless you put something else in it it does not provide su±cient sure ground for a solution. So far, axiomatic characterizations in the NTU domain have come (if at all) only after the proposals of solutions satisfying the aforementioned double consistency with Nash and Shapley (e.g., Aumann (1985) and Hart (1985)). But here integration has been accomplished directly at the axiomatic level with surprising results. Nevertheless the credit of this particular solution as characterized by Theorem 2 depends on the compellingness of the transfer condition.

In order to justify this particular solution we have provided two extensions (Theorems 3 and 4) of the solutions characterized by Theorem 1 based on two di®erent informational environments admitting random voting rules. Theorem 6 proves these extensions incompatible, but Theorem 5 proves them to be compatible within the subdomain of deterministic voting rules and coincident for TU-problems in a single case: the one characterized by Theorem 2, singling out the solution $Nash^{Sh(W)}(B)$. Is this enough to consider it as the best answer to the main question that motivated this paper? It seems clear that neither of the two scenarios considered separately provides unequivocal support for a unique answer, but we ¯nd the double consistency of this solution along with its uniqueness in the sense established in Theorem 5 rather compelling in support of this solution. It turns out to be the only one consistent with two di®erent informational environments when either the rule is deterministic or the problem TU.

Several lines of further research can be suggested. First, and possibly most important, the 'Nash program' challenge: the noncooperative foundation of the cooperative solutions characterized. Rubinstein (1982) and Binmore (1987) seem the natural term of reference as a starting point, but the achievement of this goal does not appear to be obvious at all. A completely di®erent line of work is open if the results in this paper are taken as a new starting point for addressing the normative issue about the 'fair' voting rule. That is, if the members of a committee in which bargaining is the usual practice represent groups of individuals of di®erent size, what voting rule is adequate?

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every winning coalition, but this means an ad hoc extension of the Shapley NTU value notion and of the very notion of the NTU game.

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8 Appendix

The basic idea of the following proof of uniqueness in Theorem 2 is similar to that of Nash in the proof of his classical result, but the possiblity of null players needs to be handled with care. To make the proof simpler we establish two lemmas which will be of use. In Lemma 4 and in the proof of the theorem we use the following notation. For any set $C \subseteq \mathbb{R}^N$, and $J \subseteq N$, C^J denotes the set $C^J := \{x \in C : x_i = 0 \text{ for all } i \in N \setminus J\}$. Also, for any $W \in \mathcal{W}$, we denote $\sup(W) = \{i \in N : i \text{ is not a null player in } W\}$.

Lemma 3 Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be a solution that satisfies IIA, then, for any two problems in $\mathcal{B}$, $B = (D, d)$ and $B' = (D', d')$, such that $d = d'$ and $D_d = D'_d$, and any $W \in \mathcal{W}$, it holds that $\Phi(B, W) = \Phi(B', W)$.

Proof. Just note that for two such problems $ch(D_d) = ch(D'_{d'})$, and this set is contained in D and in D'. By the individual rationality condition embodied in the solution concept, $\Phi(B, W)$ and $\Phi(B', W)$ must lie on $D_d \subseteq ch(D_d)$. Then by IIA we have $\Phi(B, W) = \Phi((ch(D_d), d), W) = \Phi(B', W)$. ■

Lemma 4 Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be a solution that satisfies $E\alpha$, An, IIA, IAT, NP and T, then, for any $(B, W) \in \mathcal{B} \times \mathcal{W}$ with $B = (D, d)$ such that $d = 0$, and $D^J = \Delta^J$ for $J = \sup(W)$, it holds that $\Phi(B, W) = Sh(W)$.

Proof. We first prove a special case. Let $W \in \mathcal{W}$, and $J = \sup(W)$. For any $\delta \in (0, 1)$, let $T^{J,\delta} : \mathbb{R}^N \to \mathbb{R}^N$ the linear map given by

$$T_i^{J,\delta}(x) := \begin{cases} x_i & \text{if } i \in J, \\ \delta x_i & \text{if } i \in N \setminus J. \end{cases}$$

Then, for any $\delta \in (0, 1)$, by IAT and Proposition 3, as $Sh_i(W) \neq 0$ if and only if $i \in J$, we have

$$\Phi(T^{J,\delta}(\cdot), W) = T^{J,\delta}(\Phi(\cdot, W)) = T^{J,\delta}(Sh(W)) = Sh(W).$$

By Lemma 3, the same conclusion holds by replacing $T^{J,\delta}(\cdot)$ by the problem $T_+^{J,\delta}(\cdot) := (T_+^{J,\delta}(\Delta), 0)$, where $T_+^{J,\delta}(\Delta) := ch(T^{J,\delta}(\Delta) \cap \mathbb{R}_+^N)$, that is, $\Phi(T_+^{J,\delta}(\cdot), W) = Sh(W)$. Now let $B = (D, d) \in \mathcal{B}$ such that $d = 0$, and $D^J = \Delta^J$ for $J = \sup(W)$. For $\delta \in (0, 1)$ sufficiently small, $T_+^{J,\delta}(\Delta) \subseteq D$. By NP, $\Phi(B, W) \in \Delta^J \cap \mathbb{R}_+^N$, and consequently $\Phi(B, W) \in T_+^{J,\delta}(\Delta)$. Then by IIA it must be $\Phi(B, W) = \Phi(T_+^{J,\delta}(\cdot), W) = Sh(W)$. ■

Proof. (of Theorem 1) Uniqueness: Let $\Phi : \mathcal{B} \times \mathcal{W} \to \mathbb{R}^N$ be a solution that satisfies $E\alpha$, An, IIA, IAT, NP and T. Let any problem (B, W), with $B = (D, d)$. Without loss of generality in view of IAT we can assume $d = 0$, and in view of Lemma 3 we can assume $D = ch(D_d)$. Denote $x^* := Nash^{Sh(W)}(B)$, which exists and is unique, and $J := \sup(W)$. By (1) and the conditions on D, it is immediate that $x_i^* \neq 0$ if and only if $i \in J$. Now let $p \in \mathbb{R}^N$ the vector

$$p_i := \begin{cases} \frac{Sh_i(W)}{x_i^*} & \text{if } i \in J, \\ 0 & \text{if } i \in N \setminus J, \end{cases}$$

and for any $k > 0$, let $B_{p,k} = (D_{p,k}, 0)$ the problem in which

$$D_{p,k} = \left\{ x \in \mathbb{R}^N : px \leq 1, \; x_i \leq k \; (\forall i \in N \setminus J) \right\}.$$

Note that $x^* \in D_{p,k}$ ($px^* = \sum_{i \in J} Sh_i(W) = 1$, and $x_i^* = 0 < k$ for all $i \in N \setminus J$)), and that $B_{p,k}$ is the result of transforming the problem $B_k = (D_k, 0)$, in which

$$D_k = \left\{ x \in \mathbb{R}^N : \sum_{i \in J} x_i \leq 1, \; x_i \leq k \; (\forall i \in N \setminus J) \right\},$$

by means of the linear map $T : \mathbb{R}^N \to \mathbb{R}^N$

$$T_i(x) := \begin{cases} \frac{1}{p_i} x_i & \text{if } i \in J, \\ x_i & \text{if } i \in N \setminus J, \end{cases}$$

that is, $B_{p,k} = T(B_k)$. As $D_k^J = \Delta^J$, by Lemma 4, $\varphi(B_k, W) = Sh(W)$. Thus by IAT

$$\varphi(B_{p,k}, W) = \varphi(T(B_k), W) = T(\varphi(B_k, W)) = T(Sh(W)) = x^*.$$

Finally, for k sufficiently large $D \cap \mathbb{R}^N_+ \subseteq D_{p,k} \cap \mathbb{R}^N_+$. Thus, again by IIA, we conclude that $\varphi(B, W) = \varphi(B_{p,k}, W) = x^*$. ■