A discusión

MIXED BUNDLING STRATEGIES AND MULTIPRODUCT PRICE COMPETITION*

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ABSTRACT

This paper deals with price competition among multiproduct firms. We consider a model with *n* firms and one representative buyer. Each firm produces a set of products that can be different or identical to the other firms' products. The buyer is characterized by her willingness to pay -in monetary terms- for every subset of products. To handle the combinatorial complexity of this general setting we use the linear relaxation of an integer programming package assignment problem. This approach allows to characterize all the equilibrium outcomes. We look for subgame perfect Nash equilibrium prices in mixed bundling strategies, i.e., when firms offer consumers the option of buying goods separately or else packages of them at a discount over the single good prices. We find that a mixed bundling subgame perfect Nash equilibrium price vector always exists. Also, the associated equilibrium outcome is always efficient, in the sense that it maximizes the social surplus. We extend the analysis to a model with *m* buyers and offer the conditions under which the equilibrium outcome set is non-empty.

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Keywords: Multiproduct price competition, Integer Programming, Mixed Bundling Strategies, Subgame Perfect Nash Equilibria.

1 Introduction

This paper deals with price competition among multiproduct firms. We consider a model with n firms and one representative buyer. Each firm produces a set of products that can be different or identical to the other firms' products. The buyer is characterized by her willingness to pay -in monetary terms- for every subset of products. We show that a mixed bundling subgame perfect equilibrium outcome always exists and it is efficient in the sense of maximizing the social surplus. Then, we extend the analysis to a model with m buyers and offer the conditions under which the equilibrium outcome set is non-empty.

Mixed bundling[1] refers to the practice of offering consumers the option of buying goods separately or else packages of them (at a discount over the single good prices). This pricing strategy has often been seen as a form of price discrimination. The traditional theory on this angle begins with the observation by Stigler (1963) that bundling can increase a seller's profits when consumers' reservation prices for two goods are negatively correlated. In the two goods case, offering both a two-good bundle as well as the individual items (mixed bundling) is typically optimal (Adams and Yellen, 1976; McAfee, McMillan and Whinston, 1989). This is because bundling reduces heterogeneity in consumer valuations, enabling a monopolist to better price discriminate (Schmalensee, 1984), while still capturing residual demand through unit sale. While the insight that bundling reduces heterogeneity in valuations is quite general, other aspects of these solutions often do not generalize beyond the two-goods case. Tractable analytical solutions have been found for a variety of special cases such as linear utilities or when valuations across different consumers can be ordered in specific ways or satisfy certain separability conditions (Armstrong, 1996; Sibley and Srinagesh, 1997). Nowadays, there have been several studies that have considered large number bundling problems in specific contexts related to information goods[2] pricing (Chuang and Sirbu, 1999). These studies generally found that engaging in a form of mixed bundling where a certain large bundle is offered along side individual sale dominates either strategy alone. However, most

[1] Examples of *mixed budling* are season tickets, film with camera, all-included vacation packages, round-trip airline tickets, etc.

[2] The emergence of Internet as a low-cost, mass distribution medium has renewed interest in pricing structures for information and other digital goods. Thus, publishers, software producers, music distributors, cable television operators, etc. face similar profits' maximizing problems.

of the results are more empirical than analytical.

The conditions of competition can be quite different when there are more than one competing firms and, therefore the analysis of the effect of mixed bundling becomes more complicated. Duopoly was considered by Economides (1993), in a model where firms produce complementary goods, and shows that mixed bundling is a dominant strategy for both firms. In a different but related set up, Liao and Urbano (2002, LU hereafter) and Liao and Tauman (2002, LT, hereafter), assume that firms produce each two complementary goods which are substitutes for the two corresponding goods produced by the other firm. Thus, the two products of each firm forms a pure system; but the firms produce modular components, in the sense that consumers can costlessly assemble mixed systems composed of any two complementary goods of the two different firms. LT find that mixed bundling strategies play a key role in stabilizing the market. If the use of mixed bundling is not allowed, LU show that subgame perfect linear pricing equilibria may fail to exists. The possibility of non existence of linear-pricing equilibrium when firms produce several goods is in contrasts with the result of Tauman *et al.* (1997, TUW, hereafter), and Arribas and Urbano (2003a, AU hereafter), who always guarantee it. These last two papers, however, deal with a simple model of price competition in a multiproduct oligopoly market, where firms only produce a product and the representative consumer buys either one or zero units of each product. In a setting of a discrete choice model of product differentiation (logit model), Anderson and Leruth (1993, AL hereafter), show that only pure component pricing (linear pricing) may be offered at equilibrium since firms fear the extra degree of competition inherent in mixed bundling. Very recently, Gandal, Markovich and Riordan (2002) have examine the importance of strategic bundling for the evolution of market structure and the performance of the PC office software market. Also through a discrete choice model of product differentiation, they find strong empirical support for negative correlation in consumer preferences over world processors and spreadsheets. This negative correlation creates an incentive for strategic bundling[3].

Most of the above models -dealing with duopoly markets, where each firm produces two complementary goods- predict mixed bundling as a result of multiproduct competition. However, a general analysis is still lacking and, what is worse, it is not even know if a Nash equilibrium may exist in such a general setting. Our analysis is a first attempt to show the ex-

[3]Incentive that Microsoft exploited succesfully with its office suite products!

istence of mixed bundling equilibrium prices in a model with multiproduct firms, where products are of a very general nature. To set up a model where oligopolistic firms may follow mixed bundling strategies, we start by assuming a representative buyer, extending the analysis to m buyers later on. Mixed bundling is an aggressive pricing policy in oligopolistic settings: it forces firms to reduce prices in an attempt to keep a competitive advantage, and therefore the assumption of a representative buyer is not as restrictive as it may appear at a first glance. Firms deal with only a (type of) buyer, what gives place to a tougher competition among firms, thus adding to the extra degree of competition inherent in mixed bundling.

Once we abandon the world of two firms and two products, the number of consumption bundles grows exponentially and it is extremely difficult to find the subgame perfect Nash equilibrium outcomes by checking and avoiding all the possible deviations. To generalize the above analysis is, then, necessary, to use tools with better handle these combinatorial complexity. In this sense, the integer programming package problem, or better, its linear relaxation allows us to characterize all price vectors satisfying Nash equilibrium subgame perfection in a huge set.

Firms do not precommit to a particular pricing strategies prior to the choice of actual prices, this meaning that mixed bundling pricing is not excluded. We look for subgame perfect Nash equilibrium prices in mixed bundling. If the use of mixed bundling strategies is not allowed, equilibrium may not exist (this was shown in LU). In contrast, we show here that a mixed bundling (subgame perfect) Nash equilibrium price vector always exists. Hence, mixed bundling is important to stabilize the market. Furthermore, we find that mixed bundling in oligopolistic competition induce consumers to select the efficient consumption set, i.e., the consumption set which maximizes the social surplus. This is not always the case, when mixed bundling is excluded.

We show that the optimal solutions of the linear relaxation of the above mentioned integer integer programming package assignment problem are the subgame perfect Nash equilibrium profits or net prices and consumption set. It is interesting to note that the optimal solutions of a linear programming problem are a polyhedron, and so is the projection of the dual problem's solutions on firms' net price vectors. This polyhedron is completely determined by its vertices. The Pareto frontier of the above projection has to be identified in order to characterize the set of *all* subgame perfect Nash equilibrium net price vectors. As this frontier can be expressed as the convex combination of non-Pareto dominated vertices, we just need to obtain all these vertices. At every equilibrium, a non-active firm sets marginal costs prices, and the active firms's net prices are the non-Pareto dominated vectors of the above Pareto frontier. When, the equilibrium consumption set is pure, i.e., it consists only of products of a firm, then non-active firms prices are set equal to marginal costs, and the selected firm's price is a bundling price, leaving the buyer with some surplus. On the contrary, if the equilibrium consumption set is composed of a product of each firm, a completely mixed bundle, then the buyer pays the sum of the individual products's prices, even though bundles are offered at special prices. Thus, mixed bundling prices are here off-equilibrium prices but are used to sustain the equilibrium. Then, LT and AL results are especial cases of our general model.

When the social value function for bundles of goods is monotonic and firms are substitutes, then equilibrium net prices are the social marginal contributions of firms, reflecting the underlying market competition. When it is convex equilibrium net prices coincide with the core of the economy.

The market model considered here is also related to the matching literature (see, Kelso and Crawford 1982, KC) and with assignment games. In particular, some extensions of the canonical standard assignment model, with many sellers and buyers interacting have received and increasing attention recently. These models are two-side matching markets, where sellers have an initial endowment of indivisible objects and buyers have an utility function over any package or bundle of objects. Differences in the framework are based on the units produced (each seller has only one product and only one unit of this product or their have no restriction on their production); the units purchased (just one or a bundle); the number of sellers (one or more); the number of buyers; the price of a bundle (additive or non-additive pricing function), etc.

The package assignment problem has been studied by Gul and Stacchetti (1999, GS), Bikhchandani and Mamer (1997, BM) and Bikhchandani and Ostroy (2001, BO), among others. In all these papers utilities are quasilinear in money, defined on bundles of goods and buyers play the same role; they select, given firms' prices, the best bundle. The main difference between our model and theirs is that we deal with strategic equilibrium where firms are price setters, while they deal with Walrasian equilibria. Other difference is that we deal with a representative buyer (although we extend some of our results to m buyers), but have no restriction on the set of goods, while, for instance, GS deal with heterogeneous buyers, but with goods which have no complementarities (a notion closely related to gross substitutability). BM and BO give a linear programming (LP) characterization of the Walrasian equilibrium outcomes while we formulate the Nash equilibrium of a multi-product market with a representative buyer, as an extension of the package assignment model and show the equivalence of some linear programming solutions and Nash equilibrium outcomes.

The paper is organized as follows. The model is presented in section 2, while the integer programming package assignment problem and its linear relaxation are offered in section 3. The main results are provided in section 4, where we show the existence of mixed bundling equilibria and characterize the equilibrium outcome set. Specific results for monotonic, concave and convex value functions are offered in section 5. Section 6 cares about the role of mixed bundling. The model with m buyers is the subject of section 7, and it concludes the paper.

2 The model

Consider an economy with n firms and one buyer. Each firm produces a set of products and one firm's products can be different from or identical to the other firms' products. Let $N = \{1, 2, ..., n\}$ be the set of firms. Let Ω_i be firm i's set of products and $\Omega = \cup_{i \in N} \Omega_i$ be the set of all products. Let $c_i(w_i)$ be the (constant) unit cost of production of firm i for product $w_i \in \Omega_i$, where costs are additive, i.e., $c_i(T_i) = \sum_{w \in T_i} c_i(w), T_i \subseteq \Omega_i$, and for any set $S \subseteq \Omega$, with $S_i = S \cap \Omega_i$ for all i, let $c(S) = (c_1(S_1), c_2(S_2), \cdots, c_n(S_n))$ be the associated cost vector.

A consumption set is a subset $S \subseteq \Omega$. A firm is said to be non-active in a given consumption set if none of its products is consumed. We will write $S_i \in S$ to mean that firm i sells set S_i in S, i.e. $S_i = S \cap \Omega_i$ and let $F(S)$ be the set of active firms in S, i.e., $F(S) = \{i \in N | S \cap \Omega_i \neq \emptyset\}$. The buyers is characterized by a value function over any subset $S \subseteq \Omega$, $v(S)$, which represents her total willingness to pay for consumption set S, with $v(\emptyset) = 0$.

Each firm i sets prices for its Ω_i products. It can also offer subsets of them as bundles for a special price. Thus, a strategy of firm i, $i \in N$, is a 2^{Ω_i}-tuple specifying the price of each $w \in \Omega_i$ as well as the prices of each any other subset of Ω_i, i.e., firm i chooses a price function $p_i \in \mathcal{P}_i$, where $\mathcal{P}_i = \mathcal{R}_+^{2^{\Omega_i}} = \{$the set of vectors $p_i : 2^{\Omega_i} \longrightarrow R_+$ with $2^{\Omega_i} = \{T_i | T_i \subseteq \Omega_i\}\}$. Let $p_i(T_i)$ be the price of $T_i \subseteq \Omega_i$, if $p_i(T_i) = \sum_{w \in T_i} p_i(w)$, then prices are linear and bundle T_i is offered for no special price. If $p_i(T_i) < \sum_{w \in T_i} p_i(w)$, then the price of T_i is subadditive, and T_i is offered as a bundle at a lower price. In this case, we say that firm i follows a *mixed bundling strategy.* To avoid irrational off-equilibrium behavior, we restrict $p_i(T_i)$, $i \in N$, $T_i \subseteq \Omega_i$, to satisfy $p_i(T_i) \geq c_i(T_i) = \sum_{w \in T_i} c_i(w)$.

The sequence of events is as follows. First, each firm i chooses a price $p_i(T_i)$ for any set $T_i \subseteq \Omega_i$ independently and simultaneously to the other firms. Then, the buyer observes price vector $p = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, and selects a consumption set $S \subseteq \Omega$ as a function of p. Formally, we have a strategic game with $n+1$ players, n firms and a representative buyer, player 0. Let $G^{MB}(n+1, v, c)$ (where MB stands for mixed bundling pricing) denote such a game. The set of strategies of each firm is the set $\mathcal{P}_i$ and that of the buyer is S_0 the set of functions $\underline{S}$ from $\mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ to 2^{Ω}. Finally, the profit function for each firm $i \in N$ is given by

$$\pi_i(\underline{S}, p) = \begin{cases} p_i(S_i) - c_i(S_i) & S_i \in \underline{S}(p) \\ 0 & S_i \notin \underline{S}(p) \end{cases}$$

where $\underline{S}(p)$ is the buyer's consumption set corresponding to p. The payoff function of the buyer is her consumer surplus: $cs(\underline{S}, p) = v(\underline{S}(p)) - \sum_{S_k \in \underline{S}(p)} p(S_k)$.

Let SPE be the set of pure strategy subgame perfect equilibria of $G^{MB}(n+1, v, c)$. If (S, p) is an element in SPE, p is called an SPE-price vector, S is an SPE-consumption set and (S, p) is denoted an SPE-outcome.

Throughout the paper we denote by $|S|$ the number of products in consumption set $S \subseteq \Omega$.

2.1 Mixed bundling pricing equilibria

Firms do not precommit to linear pricing and then the price of a subset of products can be different from the sum of the prices of its products. That is, let $p_i(T_i)$ be the price of bundle $T_i \subseteq \Omega_i$, then $p_i(T_i) \leq \sum_{w \in T_i} p_i(w)$, i.e. bundle T_i might be offered for special price. In the sequel, we characterize the SPE-outcomes, where firms might use mixed bundling strategies. Assume that the buyer can buy at most one package or bundle from each firm.

Let set $S \subseteq \Omega$ be a consumption set and recall that $F(S)$ is the set of active firms in S. The subgame perfect Nash-equilibrium conditions preclude unilateral deviations from the buyer and from each firm. Namely, we need conditions that guarantee that each active firm does not have an incentive to either increase the equilibrium prices of its sold bundles (FC1) or to modify those of unsold bundles in order to profitably sell any of them (FC2). More precisely, in a mixed bundling pricing framework the price of each bundle can be set independently of those of the other bundles with some products in common. Thus, let $(\widetilde{S}, \widetilde{p})$ be an SPE-outcome, condition FC2 below says that no firm j in $F(\widetilde{S})$ benefits from price reductions of unsold bundles: i.e. $\widetilde{S}$ has to remain a buyer best choice even if $j \in F(\widetilde{S})$ reduces the prices of every $S_j \subseteq \Omega_j$, $S_j \neq \widetilde{S}_j$, to $\widetilde{p}_j(S_j) - c_j(S_j) = \widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j)$, or $\widetilde{p}_j(S_j) = \widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j)$. The intuition is as follows. Let $j \in F(\widetilde{S})$, and consider any other consumption set S, where $j \in F(S)$. Since $(\widetilde{S}, \widetilde{p})$ is an SPE-outcome, the buyer maximizes her surplus, i.e.,

$$v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) \geq v(S) - \sum_{i \in F(S)} \widetilde{p}_i(S_i)$$

Then, firm j may have an incentive to change the price of $S_j \in S$ in order S becomes as attractive as $\widetilde{S}$ to the buyer and to obtain a profit at least equal to $\widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j)$. The minimum price verifying these properties is precisely, $p_j(S_j) = \widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j)$.

It is not difficult to show that $(\Omega, \widetilde{p})$ is an SPE-outcome iff $\widetilde{p} \geq c$ and

(BC) *Buyer optimality:* $v(\Omega) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) \geq v(S) - \sum_{i \in F(S)} \widetilde{p}_i(S_i)$, for all $S \subseteq \Omega$;

(FC1) *Firm optimality*: For every $j \in N$ there exists $S^j \subseteq \Omega \backslash \Omega_j$ such that

$$v(\Omega) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) = v(S^j) - \sum_{i \in F(S^j)} \widetilde{p}_i(S_i^j)$$

(FC2) *Firm optimality*: For each $j \in N$ and all $S \subseteq \Omega$ such that $j \in F(S)$

$$v(\widetilde{S}) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) \geq v(S) - [\widetilde{p}_j(\Omega_j) - c_j(\Omega_j) + c_j(S_j)] - \sum_{i \in F(S) \backslash j} \widetilde{p}_i(S_i).$$

Notice that (BC) is implied by the subgame perfection requirement, and (FC1) and (FC2) by firms' incentives. To see this, suppose that (FC1) does not hold, then by (BC) there exists $j \in N$, such that for all $S^j \subseteq \Omega \backslash \Omega_j$

$$v(\Omega) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) > v(S^j) - \sum_{i \in F(S^j)} \widetilde{p}_i(S_i^j)$$

and then firm j is better off charging a price $\widetilde{p}_j(\Omega_i) + \varepsilon$, for a sufficiently small $\varepsilon > 0$, such that (BC) is still satisfied for all $S^j \subseteq \Omega \backslash \Omega_j$. This implies that the buyer observing the new price vector will again choose the consumption set Ω, but firm j obtains an extra gain of ε. Hence (FC1) must be verified if $(\Omega, \widetilde{p})$ is an SPE-outcome.

If (FC2) does not hold, then for some firm $j \in N$ there exists a consumption set $S \subseteq \Omega$ such that

$$v(\widetilde{S}) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) < v(S) - [\widetilde{p}_j(\Omega_j) - c_j(\Omega_j) + c_j(S_j)] - \sum_{i \in F(S) \backslash j} \widetilde{p}_i(S_i)$$

Hence firm j can set a price $p_j(S_j) = \widetilde{p}_j(\Omega_j) - c_j(\Omega_j) + c_j(S_j) + \varepsilon$, for a sufficiently small $\varepsilon > 0$, such that

$$v(\widetilde{S}) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) < v(S) - p_j(S_j) - \sum_{i \in F(S) \backslash j} \widetilde{p}_i(S_i)$$

which implies that the buyer will select the consumption set S and firm j will increase its profits.

Conversely if (BC), (FC1) and (FC2) are satisfied then $(\Omega, \widetilde{p})$ is an SPE-outcome since Ω is a best choice for the buyer and no firm has an incentive to either reduce or increase its prices. Notice that the set S^j in (FC1) may be empty and in this case $v(\Omega) - \sum_{i \in N} \widetilde{p}_i(\Omega_i) = 0$, and firms extract the entire consumer surplus.

Suppose now that $(\widetilde{S}, \widetilde{p})$ is an SPE-outcome with $\widetilde{S} \neq \Omega$. Then, the equilibrium conditions have to additionally guarantee that no firm j outside of $\widetilde{S}$ benefits from price reductions and thus $\widetilde{S}$ has to remain a buyer best choice even if $j \notin F(\widetilde{S})$ reduces its prices to its marginal cost levels, $\widetilde{p}_j(T_j) = c_j(T_j)$, for all $T_j \subseteq \Omega_j$. The next Proposition characterizes the set of SPE-outcomes.

Proposition 1 *$(\widetilde{S}, \widetilde{p})$ is an SPE-outcome, where $\widetilde{S} \subseteq \Omega$ and $\widetilde{p} = (\widetilde{p}_1, ..., \widetilde{p}_n)$, $\widetilde{p}_i \in \mathcal{P}_i$ with $\widetilde{p} \geq c$, if and only if*

(BC) $v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) \geq v(S) - \sum_{i \in F(S)} \widetilde{p}_i(S_i)$, for all $S \subseteq \Omega$,

(FC1) For every $j \in F(\widetilde{S})$ there exists $S^j \subseteq \Omega \backslash \Omega_j$ such that

$$v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) = v(S^j) - \sum_{i \in F(S^j)} \widetilde{p}_i(S_i^j)$$

(FC2) For each $j \in F(\widetilde{S})$ and all $S \subseteq \Omega$ such that $j \in F(S)$

$$v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) \geq v(S) - [\widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j)] - \sum_{i \in F(S) \backslash j} \widetilde{p}_i(S_i)$$

(FC3) For each $j \notin F(\widetilde{S})$ and for all $S \subseteq \Omega$ such that $j \in F(S)$

$$v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) \geq v(S) - c_j(S_j) - \sum_{i \in F(S) \backslash j} \widetilde{p}_i(S_i)$$

However, we consider only the set of pure strategy subgame perfect equilibrium points of the above economy which remains as equilibrium outcomes even if *all* non-active firms set marginal cost prices and *all* active firms set prices for their unsold bundles equal to those of their sold ones adjusted by the cost-differential[4]. In other words, we want (FC3) to be satisfied for all subsets $A \subseteq N \backslash F(\widetilde{S})$ and (FC2) for all subsets $B \subseteq F(\widetilde{S})$. This restriction removes the set of equilibrium outcomes in which some firms charge unreasonably high prices so that no individual firm can benefit from a price reduction of its products only. To see this, consider the following example:

Example. Let $N = \{1, 2\}$ and $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$. Assume for simplicity that $c_i(w) = 0$ for all $i \in N, w \in \Omega$. The buyer value function is,

$$v(S) = \begin{cases} 2 & S = \{a, b\} \\ 9 & S = \{a, d\} \\ 5 & S = \{b, c\} \\ 3 & S = \{c, d\} \\ 0 & \text{otherwise} \end{cases}$$

The pair $(\widetilde{S}, \widetilde{p})$, where $\widetilde{S} = \{b, c\}$, $p_a = p_d = p_{ab} = p_{cd} = 1000$, $p_b = 3$ and $p_c = 2$, is an SPE-outcome (verifies BC to FC3), where the consumer surplus is $cs(\widetilde{S}) = v(b, c) - p_b - p_c = 0$ and profits of firms 1 and 2 are $p_b = 2$ and $p_c = 3$ respectively. However, if firm 1 reduces the prices of its unsold bundles ($\{a\}$, $\{a, b\}$) to be equal to $p_b = 3$, *and* firm 2 sets prices for bundles ($\{d\}$, $\{b, c\}$), bounded above by $p_c = 2$, then the above SPE is upset, since now $cs(a, d) = v(a, d) - p_a - p_d = 9 - 3 - 2 = 4 > 0 = cs(\widetilde{S})$ and the consumer will choose bundle $S = \{a, d\}$ instead of $\widetilde{S}$. It is easily checked that under this restriction, an SPE-outcome is $(\widehat{S}, \widehat{p})$, where $\widehat{S} = \{a, d\}$, $p_a = p_{ab} = 4$, $p_b = 0$ and $p_c = p_d = p_{cd} = 5$.

To define this restriction on the set of subgame perfect Nash equilibria consider price vector $p = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, and let $S \subseteq \Omega$. Define

[4]Notice that if marginal costs are assumed to be zero, these prices amount to be equal to those of the sold bundles.

vector p^S for all $i \in N$, $T_i \subseteq \Omega_i$, as

$$p_i^S(T_i) = \begin{cases} p_i(S_i) & \text{if } i \in F(S),\ T_i = S_i \\ p_i(S_i) - c_i(S_i) + c_i(T_i) & \text{if } i \in F(S),\ T_i \neq S_i \\ c_i(T_i) & \text{if } i \notin F(S) \end{cases}$$

i.e. the non-active firms set prices equal to the marginal cost, and all active firms set prices for unsold bundles equal to those of their sold bundles adjusted by the cost-differentials.

Definition 1 *For every triple (N, v, c) define*

$$SPE^* = \{(S,p) \in SPE | p \geq c \text{ and } (S, p^S) \in SPE\}$$

Equivalently, SPE^* is the set of equilibrium outcomes satisfying BC, FC1, and FC4 (instead of FC2 and FC3), where FC4 says,

(FC4) For all $A \subseteq N \backslash F(\widetilde{S})$, all $B \subseteq F(\widetilde{S})$ and for all $S \subseteq \Omega$ such that $(A \cup B) \subseteq F(S)$,

$$\begin{aligned} v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) \quad \geq \quad & v(S) - \sum_{i \in A} c_i(S_i) - \sum_{i \in B} [\widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) + c_i(S_i)] \\ & - \sum_{i \in F(S) \backslash (A \cup B)} \widetilde{p}_i(S_i) \end{aligned}$$

Thus, we restrict the analysis to a certain subset SPE^* of SPE-outcomes.

Notice that an SPE^*-outcome is a vector of prices and an assignment of firms to the buyer, such that each active firm sells a bundle to the buyer; firms maximize their profits and the buyer maximizes her surplus. This is quite similar to a package assignment model, where firms set prices to sell packages from among their feasible sets in order to maximize their profits.

Let $(v-c)(S) = v(S) - \sum_{i \in F(S)} c_i(S_i)$ be the *social surplus function* of the economy and let $\sum_{i \in F(S)} (p_i - c_i)(S_i) = \sum_{i \in F(S)} [p_i(S_i) - c_i(S_i)]$ be the sum of firms' profits. Define $V(K)$ as the maximum gain available in the economy consisting of a representative buyer and the firms in $K \subseteq N$, i.e. the maximum social surplus. Finally, let (S,p) be any assignment such that $S_i = \emptyset$ for all $i \in N \backslash K$ and $p_i \geq c_i$ for all $i \in K$, then

$$\begin{aligned} V(K) &= \max_{(S)} \left\{ [v(S) - \sum_{i \in F(S)} p_i(S_i)] + \sum_{i \in F(S)} [p_i(S_i) - c_i(S_i)] \right\} \\ &= \max_{(S)} \left\{ v(S) - \sum_{i \in F(S)} c_i(S_i) \right\} = \max_{(S)} \{(v-c)(S)\} \end{aligned}$$

If $\widetilde{S} \in \arg\max_{(S)}\{(v-c)(S)\}$, we say that $\widetilde{S}$ is *socially efficient*. Since we are interested in the efficiency of SPE^*-outcomes, we will compare them with the core of the economy.

Definition 2 (T-Core) *Let $T \subseteq N$. The T-core of the economy $G(n+1, v, c)$, denoted T-core(G), is the set of $(n+1)$-tuples $(q^b, \{q_i\}_{i\in N}) \in R_+^{n+1}$, such that*
(i) $q^b + \sum_{i\in N} q_i = V(T)$
(ii) $q^b + \sum_{i\in K} q_i \geq V(S), \qquad \forall S \subseteq N$

The element q^b is the consumer surplus and each q_i are firm i's profits. For $T = N$, we obtain the core of the economy, denoted by core(G). Also, the subset of points in T-core(G) such that the buyer surplus, q^b, is equal to zero defines the T-core of $(v-c)$ or T-core$(v-c)$,

$$T\text{-core}(v-c) = \{q \in R_+^n | \sum_{i\in T} q_i = V(T) \text{ and } \sum_{i\in S\cap T} q_i \geq V(S) \text{ for all } S \subseteq N\}$$

The intuition of T-core$(v-c)$ is as follows. Suppose that the equilibrium consumption set is T. Then, firms in $N\backslash T$ obtain zero profits and hence will be willing to join the set of seller firms. Hence, every subset $S \subseteq T$ can actually achieve $u_T(S)$ where,

$$u_T(S) = \max_{A\subseteq N\backslash T}\{(v-c)(S\cup A)\}$$

It is easily checked that the projection of the T-core$(v-c)$ on T coincides with the N-core(u_T), or core$(v-c)$, when the buyer surplus is zero.

Finally, we assume that the social surplus of the economy is positive. Otherwise, if for every consumption set S, $(v(S) - \sum_{i\in F(S)} c_i(S_i)) < 0$, then the economy is degenerated. Hence, at every equilibrium point (S, p), $S = \emptyset$ must hold and therefore, no production will take place.

2.2 Examples

Example 1: Let $N = \{1, 2\}$ and $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$. Assume for simplicity that $c_i(w) = 0$ for all $i \in N, w \in \Omega$. The buyer value function is,

$$v(S) = \begin{cases} 16 & S = \{a, b\} \\ 15 & S = \{a, d\} \\ 14 & S = \{b, c\} \\ \delta & S = \{c, d\} \\ 0 & \text{otherwise} \end{cases}$$

Table 1 offers the set of SPE and SPE^* -outcomes. The two sets coincide because condition FC3 gives the lower bounds of prices in both sets, and FC4 is not binding for prices and only implies efficiency of $\widetilde{S}$ .The equilibrium consumption set is always efficient and firm 2 sets prices equal to marginal costs (which here are zero).

Table 1: $SPE = SPE^*$-outcomes of example 1

$0 < \delta \leq 13$	$13 < \delta < 16$
$\widetilde{S} = \{a, b\}$, efficient	$\widetilde{S} = \{a, b\}$, efficient
$p_1^a \geq 15 - \delta$	$p_1^a \geq \max\{15 - \delta, 0\}$, $p_1^b \geq \max\{14 - \delta, 0\}$
$p_1^b \geq 14 - \delta$	$p^S = \{p_1^a = p_1^b = p_1^{ab} = 16 - \delta,\ p_2^c = p_2^d = p_2^{cd} = 0\}$
$p_1^{ab} = 16 - \delta$	$p_1^{ab} = 16 - \delta$
$p_2^c \geq 0$	$p_2^c \geq 0$
$p_2^d \geq 0$	$p_2^d \geq 0$
$p_2^{cd} = 0$	$p_2^{cd} = 0$
$cs = \delta$	$cs = \delta$

The assumption that the consumer can buy at most a bundle from each firm may imply superadditive prices; for instance, if we consider only the prices' lower bounds $p_1^{ab} < p_1^a + p_1^b$, for $0 < \delta < 13$, $p_1^{ab} = p_1^a + p_1^b$, $\delta = 13$ and $p_1^{ab} > p_1^a + p_1^b$, for $13 < \delta < 16$. However, this possibility is ruled out for SPE^*-outcomes under vector p^S, where $p_1^a = p_1^b = p_1^{ab} = 16 - \delta$ for firm 1 and $p_2^c = p_2^d = p_2^{cd} = 0$, for firm 2. Firm 1's profits are $p_1^{ab} = 16 - \delta$, firm 2's profits are zero and the consumer surplus is $cs = \delta$.

The core of the economy is,

$$core(G) = \{(q^b, q_1, q_2) \in R_+^3 | 0 \leq q_1 \leq 16 - \delta,\ q_2 = 0,\ q^b = 16 - q_1 \geq \delta\}.$$

Hence, the SPE^*-price vector is the element of core(G) which maximizes the firms' profits.

Example 2: Let $N = \{1, 2\}$ and $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$, $c_i(w) = 0$ for all $i \in N, w \in \Omega$, and let the buyer value function be,

$$v(S) = \begin{cases} 6 & S = \{a, b\} \\ 9 & S = \{a, d\} \\ \delta & S = \{b, c\} \\ 7 & S = \{c, d\} \\ 0 & \text{otherwise} \end{cases}$$

Table 2a shows the set of SPE-equilibrium outcomes, while table 2b offers the SPE^* mixed bundling outcomes.

Table 2a: SPE-equilibrium outcomes of example 2

$0 < \delta < 9$	$0 < \delta < 9$	$7 \leq \delta < 9$
$\widetilde{S} = \{c, d\}$, inefficient	$\widetilde{S} = \{a, d\}$, efficient	$\widetilde{S} = \{b, c\}$, inefficient
$p_1^a \geq 2$	$p_1^a \leq 2$	$p_1^a \geq p_1^b + 9 - \delta$
$p_1^b \geq \max\{\delta - 7, 0\}$	$p_1^b \geq \max\{p_1^a + \delta - 9, 0\}$	$p_1^b \leq \delta - 7$
$p_1^{ab} = 0$	$p_1^{ab} = p_1^a + p_2^d - 3$	$p_1^{ab} = p_1^b + p_2^c + 6 - \delta$
$p_2^c \geq \max\{\delta - 6, 0\}$	$p_2^c \geq \max\{p_2^d + \delta - 9, 0\}$	$p_2^c \leq \delta - 6$
$p_2^d \geq 3$	$p_2^d \leq 3$	$p_2^d \geq p_2^c + 9 - \delta$
$p_2^{cd} = 1$	$p_2^{cd} = p_1^a + p_2^d - 2$	$p_2^{cd} = p_1^b + p_2^c + 7 - \delta$
		$p_1^b + p_2^c \geq \delta - 6$
$cs = 6$	$cs = 9 - p_1^a - p_2^d$	$cs = \delta - p_1^b - p_2^c$

Table 2b: SPE^*-outcomes

$0 < \delta < 9$
$\widetilde{S} = \{a, d\}$, efficient
$p_1^a = 2$
$p_1^b \geq \max\{\ \delta - 7,\ 0\}$
$p_1^{ab} = 2$
$p_2^c \geq \max\{\ \delta - 6,\ 0\}$
$p_2^d = 3$
$p_2^{cd} = 3$
$p_1^b + p_2^c \geq \max\{\delta - 4, 0\}$
$cs = 4$

Thus, firm 1's profits under SPE^*-outcomes are $p_1^a = 2$, firm 2's profits are $p_2^d = 3$ and the consumer surplus is $cs = 4$. Notice that, SPE-outcomes, need not be efficient. For instance, $((c, d), (2, 2, 0), (3, 3, 1)) \in SPE$-outcome set, but it is socially inefficient given that $v(c, d) < v(a, d)$.

Notice that SPE^*-prices imply $p_1^{ab} \leq p_1^a + p_1^b$ and $p_2^{cd} \leq p_2^c + p_2^d$, for all δ. However, although firms offer their two products as a bundle for a special price, the buyer selects a product of each firm. Here, mixed bundling is an off-equilibrium pricing strategy, supporting equilibrium outcomes. The core of the economy is given by the following set,

$$core(G) = \{(q^b, q_1, q_2) \in R_+^3 | 0 \leq q_1 \leq 2, 0 \leq q_2 \leq 3, q^b = 9 - q_1 - q_2 \geq 4\}$$

Thus, as above, the vector of prices of SPE^*-outcomes is the element of core(G) which maximizes the firms' profits.

3 The associated Package Assignment problem.

Our main result shows that there always exists an equilibrium outcome in our model. Moreover, the set of SPE^*-outcomes is equivalent to integer-valued solutions of the linear relaxation of a package assignment problem (**LP** hereafter).

For any $S \subseteq \Omega$, define z_S which is equal to 1 if the buyer chooses consumption set S; and for all firm $i \in N$ and any set of its products $T_i \subseteq \Omega_i$, let $y(T_i, i) = 1$ if firm i sells bundle T_i, and zero otherwise. The integer programming defining the package assignment problem, denoted **ILP** is,

$$
\begin{aligned}
V(\Omega) = Max \quad & \sum_{S \subseteq \Omega} (v-c)(S) z_S \\
s.t. \quad & \sum_{S \subseteq \Omega} z_S \leq 1 && (1) \\
& \sum_{T_i \subseteq \Omega_i} y(T_i, i) \leq 1 \qquad \forall i \in N && (2) \\
& \sum_{S \ni T_i} z_S \leq y(T_i, i) \qquad \forall i \in N, \forall T_i \subseteq \Omega_i && (3) \\
& z_S,\ y(T_i, i) \in \{0, 1\} \qquad \forall i \in N, \forall T_i \subseteq \Omega_i, \forall S \subseteq \Omega
\end{aligned}
$$

The first constraint ensures that only one consumption set is selected. The other constraints are redundant given the first one, in the sense that they do not reduce the set of feasible solutions. However, constraints (2) and (3) will define the price vector in the dual problem. Constraints in (2) guarantee that each firm only sells one consumption set, and constraints (3) ensures that firm i sells $T_i \subseteq \Omega_i$ if and only if the selected consumption set S is such that $S_i = T_i$.

Let us consider the linear relaxation **LP** of **ILP** in which we change the integrity constraints z_S, $y(T_i, i) \in \{0, 1\}$ in **ILP** to $z_S \geq 0$, $y(T_i, i) \geq 0$. Let **DLP** be the dual of **LP.** The interest of this formulation is that each dual variable associated with each constraint in (2) can be interpreted as firm i's profits (the buyer's payment for T_i to firm i minus its marginal cost) and each dual variable associated with constraints (3), as the net price that firm i sets for each $T_i \subseteq \Omega_i$ (i.e. prices minus marginal costs). More precisely, to write the dual problem, we associate a variable with each of the constraints of **LP**. Let π^b -the consumer surplus- be the variable associated to the first constraint ; let π_i be the ones associated to constraints (2) and finally, let

$\pi^i_{S_i}$ be those associated with constraints (3). The dual problem, **DLP**, is

$$\begin{array}{lll} Min & \pi^b + \sum_{i \in N} \pi_i & \\ s.t. & \pi^b + \sum_{S_i \in S} \pi^i_{S_i} \geq (v-c)(S) \qquad \forall S \subseteq \Omega & (4) \\ & \pi_i - \pi^i_{T_i} \geq 0 \qquad \forall i \in N,\ \forall T_i \subseteq \Omega_i & (5) \\ & \pi^b, \pi_i, \pi^i_{T_i} \geq 0 & \end{array}$$

Let $(\pi^b, (\pi_i), (\pi^i_{T_i}))$ express a generic solution of **DLP**. The set of solutions of **ILP** is the set of optimal feasible solutions (vertex points) of **LP** because of its special structure. Notice that if we remove the redundant constraints of **LP** we are left with the constraint whose coefficients are equal to 1 and the non-negativity conditions on variables S, for all $S \subseteq N$. It is well known that the solutions for such a problem are integer: the variable corresponding to the maximum coefficient in the objective function is set to 1 and the remaining variables are set to 0. Hence, in our case, an integer solution always exists and it is the consumption set $\widetilde{S} \subseteq \Omega$ such that $\widetilde{S} \in \arg\max_{S \subseteq \Omega}(v-c)(S)$.

Moreover, by the fundamental duality theorem (see Dantzig, 1974, p.125), if the primal problem has an optimal feasible solution, so does its dual problem and the two optimal value functions are the same. Also notice that the set of solutions is a convex polyhedron. Denote this set by $sol(.)$.

Interpreting variables (π_i) of the dual problem as firms' profits, let us define

$$\Pi = \{(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{DLP}) | \text{ there is no other } (\pi'^b, (\pi'_i), (\pi'^i_{T_i})) \in sol(\mathbf{DLP}), \\ \text{such that } \pi'_i \geq \pi_i, \text{ for all } i \text{ and } \pi'_j > \pi_j \text{ for at least some } j\}$$

as the Pareto frontier of set $sol(\mathbf{DLP})$. We will see below that the projection of Π on coordinates (π_i) will provide the firms' equilibrium profits. Furthermore, the set Π can be expressed as the convex combination of adjacent vertices. A way to obtain some of these vertices is to consider, among all solutions of the dual problem, those maximizing $\sum_{i \in N} \pi_i$. Namely, consider the restricted dual problem, **RDLP**,

$$\begin{array}{ll} Max & \sum_{i \in N} \pi_i \\ s.t. & \pi^b + \sum_{S_i \in S} \pi^i_{S_i} \geq (v-c)(S) \qquad \forall S \subseteq \Omega \\ & \pi_i - \pi^i_{T_i} \geq 0 \qquad \forall i \in N, \ \forall T_i \subseteq \Omega_i \\ & \pi^b + \sum_{i \in N} \pi_i = V(\Omega) \\ & \pi^b, \pi_i, \pi^i_{T_i} \geq 0 \end{array} \tag{6}$$

where the first and second restrictions are (4) and (5) in **DLP**.

To generate all the frontier Π we define a family of problems which take into account the lexicographic order of the solutions of **DLP.** To this end, let μ be an ordered partition of N in the sense that the order of the elements in the partition is relevant. Thus, μ and μ' can give rise to the same partition, but with a different order in their elements. Let Γ denote the set of all the ordered partitions. Write $\mu = \{N_1, N_2, ..., N_L\} \in \Gamma$ to mean that under μ the first element of the partition is N_1, the second in N_2 and the last one in N_L. Note that L can differ from one partition to another.

The dual problem under this partition-approach, μ-**DLP**, is then

$$\begin{array}{ll} Max & \sum_{l=1}^{L} \left(\sum_{i \in N_l} \pi_i \right) 10^{d(L-1)} \\ s.t. & \pi^b + \sum_{S_i \in S} \pi^i_{S_i} \geq (v-c)(S) \qquad \forall S \subseteq \Omega \\ & \pi_i - \pi^i_{T_i} \geq 0 \qquad \forall i \in N, \ \forall T_i \subseteq \Omega_i \\ & \pi^b + \sum_{i \in N} \pi_i = V(\Omega) \\ & \pi^b, \pi_i, \pi^i_{T_i} \geq 0 \end{array}$$

where d is an integer such that $Card(N) \cdot (v-c)(S) < 10^d$ for all $S \subseteq \Omega$. Note that for $\mu = \{N\}$ we have μ-**DLP** = **RDLP**.

The partition formulation does not change the constraints but makes the objective function vary. The objective function is an integer for which each set of d (consecutive) digits are determined by $\sum_{i \in N_l} \pi_i$. Thus, the first d digits are *occupied* by $\sum_{i \in N_1} \pi_i$, the second d digits by $\sum_{i \in N_2} \pi_i$

and so on, and, finally, the last d digits by $\sum_{i\in N_L}\pi_i$. In this way, sol(μ-**DLP**) $\subseteq$ sol(**DLP**) so that, a solution in μ-**DLP** gives one of the most preferred profit vectors by firms in N_1; it gives one of the most preferred profit vectors by the set of firms in N_2, among those most preferred by firms in N_1; and so on.

Also notice that if $(\widetilde{\pi}^b,(\widetilde{\pi}_i),(\widetilde{\pi}^i_{T_i}))\in sol(\mu-\mathbf{DLP})$, then so does $(\widetilde{\pi}^b,(\widetilde{\pi}_i),(\pi^i_{T_i}))$, where $\widetilde{\pi}^i_{T_i}\leq\pi^i_{T_i}\leq\widetilde{\pi}_i$ for all $i\in N, T_i\subseteq\Omega_i$. As we will see below, each variable $\pi^i_{T_i}$ of the dual problem defines firm i's net price. Moreover, the $\mu-$**DLP** approach does not assume any restriction on variables $(\widetilde{\pi}^i_{T_i})$ for all $i\in N, T_i\subseteq\Omega_i$, i.e., there is not restriction on the relationship between $\widetilde{\pi}^i_{T_i}$ and $\sum_{w\in T_i}\widetilde{\pi}^i_w$. This translates to firm i's pricing strategies, so that mixed bundling strategies are allowed. In fact, the solution in which $\pi^i_{T_i}=\widetilde{\pi}_i$ for all $i\in N, T_i\subseteq\Omega_i$ will define a mixed bundling price equilibrium.

The next Lemma shows that the dual solutions achieved by different partitions are in set Π.

Lemma 1 *Let $\mu\in\Gamma$ and let $(\pi^b,(\pi_i),(\pi^i_{T_i}))\in sol(\mu$-**DLP**$)$, then $(\pi^b,(\pi_i),(\pi^i_{T_i}))$ belong to Π.*

Proof: If $(\pi^b,(\pi_i),(\pi^i_{T_i}))\notin\Pi$, then there exists $(\widetilde{\pi}^b,(\widetilde{\pi}_i),(\widetilde{\pi}^i_{T_i}))\in\Pi$ such that $\pi_i\leq\widetilde{\pi}_i$ for all $i\in N$ and $\pi_k<\widetilde{\pi}_k$, for some $k\in N$. Let $\mu=\{N_1,N_2,...,N_L\}$ then,

$$\begin{aligned}F(\mu,\pi) &= \sum_{l=1}^{L}\left(\sum_{i\in N_l}\pi_i\right)10^{d(L-l)}\\ &< \sum_{l=1}^{L}\left(\sum_{i\in N_l}\widetilde{\pi}_i\right)10^{d(L-l)}=F(\mu,\pi')\end{aligned}$$

which implies that $(\pi^b,(\pi_i),(\pi^i_{T_i}))$ is not a solution of μ-**DLP**, a contradiction. ■

Define the binary relation *to be coarser than* in set Γ as follows. Given $\mu,\mu'\in\Gamma$, we say that $\mu'=\{N'_1,...,N'_L\}$ is coarser than $\mu=\{N_1,...,N_M\}$ if

$$\begin{aligned}N'_1 &= N_1\cup N_2\cup...\cup N_{n_1}\\ N'_2 &= N_{n_1+1}\cup...\cup N_{n_2}\\ \vdots &= \vdots\\ N'_L &= N_{n_{L-1}}\cup...\cup N_M\end{aligned}$$

Clearly this binary relation is reflexive, anti-symmetric and transitive so that it induces a partial order relation, with all the maximal chains ending in N and starting in any of the total partitions of N.

Coarser partitions have more degrees of freedom and hence the sum of dual solutions $\{\pi_i\}_{i\in N}$ is bigger. This is proven in the following Lemma.

Lemma 2 *Let $\mu, \mu' \in \Gamma$, with μ coarser than μ'. Then $\pi^b \leq \widetilde{\pi}^b$, for all $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mu\text{-}\mathbf{DLP})$ and $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mu'\text{-}\mathbf{DLP})$.*

Proof: It suffices to prove it for two consecutive partitions of a maximal chain, $\mu = \{N_1, ..., N_l, ..., N_L\}$, $\mu' = \{N_1, ..., N_{l_1}, N_{l_2}, ..., N_L\}$, where $N_l = N_{l_1} \cup N_{l_2}$.

Clearly, $\sum_{i\in N_k} \pi_i = \sum_{i\in N_k} \widetilde{\pi}_i$ for $k = 1, ..., l-1$. Moreover, $\sum_{i\in N_l} \pi_i \geq \sum_{i\in N_{l_i}} \widetilde{\pi}_i + \sum_{i\in N_{l_2}} \widetilde{\pi}_i$ and $\sum_{N_{l+1}\cup...\cup N_L} \pi_i \geq \sum_{NS_{l+1}\cup...\cup N_L} \widetilde{\pi}_i$.

Hence, $\sum_{i\in N} \pi_i \geq \sum_{i\in N} \widetilde{\pi}_i$which implies that $\pi^b \leq \widetilde{\pi}^b$. ■

EXAMPLE 3: Let $N = \{1, 2\}$, $\Omega_1 = \{a, b\}$, $\Omega_2 = \{c, d\}$, with $c_i(w) = 0$ for all $i \in N, w \in \Omega$, and let the value function be,

$$v(S) = \begin{cases} 6 & S = \{a, b\} \\ 15 & S = \{a, d\} \\ 14 & S = \{b, c\} \\ 8 & S = \{c, d\} \\ 0 & \text{otherwise} \end{cases}$$

The full list of solutions of μ-**DLP** problems is:

- $\mu = \{\{1\}, \{2\}\}$: $\pi^b = 0, \pi_1 = \pi^1_a = \pi^1_{ab} = 7, \pi^1_b = 6$ and $\pi_2 = \pi^2_c = \pi^2_d = \pi^2_{cd} = 8$, which maximizes firm 1's profit and yields, jointly with the primal solution, outcome $(\widetilde{S}, \widetilde{p})$:

$$\begin{aligned} \widetilde{S} &= \{a, d\} \\ \widetilde{p}_1 &= (\widetilde{p}^a_1 = 7, 6 \leq \widetilde{p}^b_1 \leq 7, \widetilde{p}^{ab}_1 = 7) \\ \widetilde{p}_2 &= (\widetilde{p}^c_2 = 8, \widetilde{p}^d_2 = 8, \widetilde{p}^{cd}_2 = 8). \end{aligned}$$

  It can be checked that $(\widetilde{S}, \widetilde{p})$ is an SPE^*-outcome

- $\mu = \{\{2\}, \{1\}\}$: $\pi^b = 0, \pi_1 = \pi^1_a = \pi^1_{ab} = 6, \pi^1_b = 5, \pi_2 = \pi^2_c = \pi^2_d = 9$ and $\pi^2_{cd} = 8$, which maximizes firm 2's profit and now yields SPE^*-

outcome $(\widetilde{S}, \widetilde{p})$:

$$\begin{aligned} \widetilde{S} &= \{a, d\} \\ \widetilde{p}_1 &= (\widetilde{p}_1^a = 6, 5 \leq \widetilde{p}_1^b \leq 6, \widetilde{p}_1^{ab} = 6) \\ \widetilde{p}_2 &= (8 \leq \widetilde{p}_2^c \leq 9, \widetilde{p}_2^d = 9, \widetilde{p}_2^{cd} = 8). \end{aligned}$$

Moreover, both are solutions of **RDLP** and any convex combination of them is also a solution of **RDLP**. In fact, given two different partitions, μ and μ', and their associated solutions, $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mu-\mathbf{DLP})$ and $(\pi'^b, (\pi'_i), (\pi'^i_{T_i})) \in sol(\mu'-\mathbf{DLP})$, either $(\pi_i) = (\pi'_i)$ or there is no Pareto dominance between them. For instance, partitions $\mu = \{\{2\}, \{1\}\}$ and $\mu' = \{N\}$ define the same optimal solution, but $\mu = \{\{2\}, \{1\}\}$ and $\mu' = \{\{1\}, \{2\}\}$ define solutions with non-Pareto dominance in components (π_1, π_2).

Thus, $(\widetilde{S}, \widetilde{p})$ is an SPE^*-outcome with $\widetilde{S} \in \arg\max_{S \subseteq \Omega}\{(v-c)(S)\}$ and price vector $\widetilde{p}$ defined as $\widetilde{p}_i(T_i) = \pi^i_{T_i} + c(T_i)$ for all firm $i \in N$ and all $T_i \subseteq \Omega_i$. Notice that firm i sets the price of any bundle $T_i \subseteq \Omega_i$, $T_i \cap \widetilde{S} = \emptyset$, lower than or equal to the price of bundle $\widetilde{S}_i$. As in examples 1 and 2 above, the payoffs of SPE^*-outcomes are the points of core(G) which maximizes firms' joint profits,

$$\begin{aligned} core(G) = \{(q^b, q_1, q_2) \in R^3_+ | 0 \leq q_1 \leq 7, 0 \leq q_2 \leq 9, q_1 + q_2 \leq 15, \\ q^b = 15 - q_1 - q_2 \geq 0\} \end{aligned}$$

4 Mixed Bundling Subgame Perfect Nash equilibria via Linear programming

4.1 Existence

Our main result establishes that an optimal solution of **LP** and $\mu-$**DLP** is an SPE^*-outcome (Proposition 2) of G^{MB}. Moreover, the SPE^*-consumption set is always efficient (Corollary 3). First, we start with a general property which states that optimal solutions of **LP** and **DLP**, set the prices of non-active firms equal to marginal costs and hence their profits are zero (property ii) and the profits of any active firm are bigger than or equal to their selling prices minus marginal costs (property i). The proofs are in the Appendix.

Lemma 3 *Let $\widetilde{S} \in sol(\mathbf{LP})$ and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{DLP})$, then* *i)* $\pi_i = \pi^i_{\widetilde{S}_i}$ $(\geq \pi^i_{T_i})$ *for all* $i \in F(\widetilde{S}), T_i \subseteq \Omega_i$

ii)$\pi_i = \pi^i_{T_i} = 0$ *for all* $i \notin F(\widetilde{S})$ *and* $T_i \subseteq \Omega_i$.

The next Proposition gives an existence result. It shows that any element of $\langle sol(\mathbf{LP}) \times sol(\mu\text{-}\mathbf{DLP})\rangle$ is an SPE^*-outcome, i.e., $sol(\mathbf{LP})$ give the equilibrium consumption set and $sol(\mu\text{-}\mathbf{DLP})$ the equilibrium profits and price vectors, for some partition μ.

Proposition 2 *Let v be a value function and c a marginal cost vector. Let $\widetilde{S} \in sol(\mathbf{LP})$ and $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mu - \mathbf{DLP})$. Then, $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set of $G^{MB}(n+1, v, c)$, where $\widetilde{\pi}^i_{T_i} + c_i(T_i) \leq \widetilde{p}_i(T_i) \leq \widetilde{\pi}_i + c_i(T_i)$ for all $i \in N$ and all $T_i \subseteq \Omega_i$ and $\widetilde{p}_i(\widetilde{S}_i) = \widetilde{\pi}^i_{\widetilde{S}_i} + c_i(\widetilde{S}_i)$.*

Let us apply the above results to the previous examples.

EXAMPLE 1 (continuation): Solving the primal and dual problems for $\mu = \{N\}$ and for different values of δ we find,

δ	$sol(\mathbf{LP})$	π^1_a	π^1_b	π^1_{ab}	π^2_c	π^2_d	π^2_{cd}
4	$\{a,b\}$	11	12	12	0	0	0
12	$\{a,b\}$	3	4	4	0	0	0
15	$\{a,b\}$	0	1	1	0	0	0

Defining $\widetilde{S} = \{a, b\}$ and $\widetilde{p}_i(T_i) = \pi^i_{T_i}$, then $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set, as we can check in the right hand side of table 1. For instance, for $\delta = 12$, the solutions to the primal and dual problems yield the SPE^*-outcome,

$$\begin{aligned} \widetilde{S} &= \{a, b\} \\ \widetilde{p}_1 &= (3 \leq \widetilde{p}^a_1 \leq 4, \widetilde{p}^b_1 = 4, \widetilde{p}^{ab}_1 = 4) \\ \widetilde{p}_2 &= (\widetilde{p}^c_2 = 0, \widetilde{p}^d_2 = 0, \widetilde{p}^{cd}_2 = 0). \end{aligned}$$

EXAMPLE 2 (continuation): proceeding similarly, we find that $sol(\mathbf{LP}) = \{a, d\}$ and $\pi^1_a = \pi^1_b = \pi^1_{ab} = 2$ and $\pi^2_c = \pi^2_d = \pi^2_{cd} = 3$ for all δ. Thus, $\widetilde{S} = \{a, b\}$, $\widetilde{p}_1(T_1) = 2$ for all $T_1 \subseteq \Omega_1$ and $\widetilde{p}_2(T_2) = 3$ for all $T_2 \subseteq \Omega_2$ define an SPE^*-outcome as we can see in the right hand side of table 2.

Finally, recalling that when $\mu = \{N\}$, μ-$\mathbf{DLP} = \mathbf{RDLP}$, we notice that the associated SPE^*-outcomes give the lowest surplus for the buyer. The proof is in the Appendix.

Corollary 1 *Let v be a value function and c a marginal cost vector, let $\widetilde{S} \in sol(\mathbf{LP})$ and let $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mathbf{RDLP})$, then among all SPE^*-outcomes, $(\widetilde{S}, \widetilde{p})$ is the one which gives the lowest surplus to the buyer, where $\widetilde{\pi}^i_{T_i} + c_i(T_i) \leq \widetilde{p}_i(T_i) \leq \widetilde{\pi}_i + c_i(T_i)$ for all $i \in N$ and all $T_i \subseteq \Omega_i$ and $\widetilde{p}_i(\widetilde{S}_i) = \widetilde{\pi}^i_{\widetilde{S}_i} + c_i(\widetilde{S}_i)$.*

Corollary 2 *For every value function v and marginal cost vector c, there always exists an SPE^*-outcome of G^{MB}.*

Proof: The existence of an optimal solution of **LP** is immediate and the fundamental duality theorem guarantees the existence of an optimal solution of the dual linear program and hence of the restricted dual problem. Finally, Proposition 2 shows that both a primal optimal solution is an equilibrium consumption set and an optimal solution of the restricted dual problem defines an equilibrium price vector. ■

Next, we show the efficiency of any equilibrium consumption set.

Corollary 3 *For every value function v and marginal cost vector c, $\widetilde{S} \subseteq \Omega$ is an equilibrium consumption set of G^{MB} if and only if $\widetilde{S}$ is socially efficient, i.e., $\widetilde{S} \in \arg\max_{S \subseteq \Omega}\{(v-c)(S)\}$.*

Proof: Let $\widetilde{S} \in \arg\max_{S \subseteq \Omega}\{(v-c)(S)\} = sol(\mathbf{LP})$ and consider any $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mathbf{RDLP})$, then by Proposition 2, $(\widetilde{S}, ((\widetilde{\pi}^1_{T_1}), ..., (\widetilde{\pi}^n_{T_n}))) \in SPE^*$-outcome set.

Now, let $(\widetilde{S}, \widetilde{p})$ be an SPE^*-outcome, we will show that $\widetilde{S} \in sol(\mathbf{LP})$. If $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set, then $(\widetilde{S}, \widetilde{p}^{\widetilde{S}}) \in SPE^*$-outcome set, where we define $\widetilde{p}^{\widetilde{S}}$ for all $i \in N$ and all $T_i \subseteq \Omega_i$, as

$$p_i^{\widetilde{S}}(T_i) = \begin{cases} p_i(\widetilde{S}_i) & \text{if } i \in F(\widetilde{S}),\ T_i = \widetilde{S}_i \\ p_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) + c_i(T_i) & \text{if } i \in F(\widetilde{S}),\ T_i \neq \widetilde{S}_i \\ c_i(T_i) & \text{if } i \notin F(\widetilde{S}) \end{cases}$$

By BC,

$$v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i^{\widetilde{S}}(\widetilde{S}_i) \geq v(S) - \sum_{i \in F(S)} \widetilde{p}_i^{\widetilde{S}}(S_i)$$

for all $S \subseteq \Omega$. Therefore,

$$\begin{aligned}(v-c)(\widetilde{S})-(v-c)(S) &\geq \sum_{i\in F(\widetilde{S})}(\widetilde{p}_i^{\widetilde{S}}-c_i)(\widetilde{S}_i)-\sum_{i\in F(S)}(\widetilde{p}_i^{\widetilde{S}}-c_i)(S_i)\\ &= \sum_{i\in F(\widetilde{S})}(\widetilde{p}_i^{\widetilde{S}}-c_i)(\widetilde{S}_i)-\sum_{i\in F(\widetilde{S})\cap F(S)}(\widetilde{p}_i^{\widetilde{S}}-c_i)(S_i)\\ &= \sum_{i\in F(\widetilde{S})\setminus F(S)}(\widetilde{p}_i-c_i)(\widetilde{S}_i)\geq 0,\end{aligned}$$

given that $\widetilde{p}_i^{\widetilde{S}}(S_i)=c_i(S_i)$ for all $i\notin F(\widetilde{S})$ and $p_i^{\widetilde{S}}(S_i)-c_i(S_i)=p_i^{\widetilde{S}}(\widetilde{S}_i)-c_i(\widetilde{S}_i)$ for all $i\in F(\widetilde{S})\cap F(S)$. Thus, $(v-c)(\widetilde{S})\geq(v-c)(S)$ for every $S\subseteq\Omega$.
■

4.2 Characterization of SPE^*-outcomes via Linear Programming

In this section we characterize the set of equilibrium prices of efficient consumption sets in G^{MB}. By Proposition 2, the optimal solutions of μ-**DLP** characterize the set of SPE^*-price vectors, p^S. But, as example 1 shows, the reverse of Proposition 2 need not be satisfied: SPE^*-outcomes does not always define a solution of the primal and the restricted dual problems. Notice, however, that if (S,p) and (S,p^S) are two SPE^*-outcomes then, clearly, the firms and the buyer obtain the same payoffs under such outcomes: the two equilibria are payoff-equivalent. Thus, any pair (S,p^S) allows to identify its payoff-equivalence class. For any set of payoff equivalent SPE^*-outcomes, we are only considering (S,p^S) as the representative outcome of this equivalence class.

Next result asserts that there is not Pareto dominance between the profits' vectors associated to two different equilibrium outcomes.

Lemma 4 *Let $(\widetilde{S},\widetilde{p})$, $(\widetilde{S},\widehat{p})$ be two SPE^*-outcomes, then there exists two active firms $j,j'\in F(\widetilde{S})$ such that $(\widetilde{p}_j-c_j)(\widetilde{S}_j)>(\widehat{p}_j-c_j)(\widetilde{S}_j)$ and $(\widehat{p}_{j'}-c_{j'})(\widetilde{S}_{j'})>(\widetilde{p}_{j'}-c_{j'})(\widetilde{S}_{j'})$.*

Proof: Suppose that $(\widetilde{p}_k-c_k)(\widetilde{S}_k)\geq(\widehat{p}_k-c_k)(\widetilde{S}_k)$ for all $k\in N$ where the inequality is strict for some firm $j\in F(\widetilde{S})$. Then, j has incentives to raise its equilibrium prices at $(\widetilde{S},\widehat{p})$, which is a contradiction. ■

We show next, that every firm's profit vector associated to an SPE^*-price vector coincides with an optimal solution of the dual problem. i.e., every SPE^*-outcome (S,p) is such that S is the solution of the primal problem and p^S defines a solution of the dual problem. Also, by Lemma 4, it does not exist any other solution of the **DLP,** $(\pi^b,(\pi_i),(\pi^i_{T_i}))$, such that its components (π_i) weakly Pareto dominates the firms' profit vector of any other SPE^*-outcome. The reverse also holds: given both an **LP** and **DLP** optimal solutions, with no weakly Pareto dominated component (π_i), they yield an SPE^*-outcome. In other words, the set of firms' profit vectors associated to SPE^*-outcomes is the Pareto frontier of the polyhedron of projection-$sol(\mathbf{DLP})$ to components (π_i).

Proposition 3 *Let v be a value function. $(\widetilde{S},\widetilde{p}^{\widetilde{S}}) \in SPE^*$-outcome set, if and only if*

i) $\widetilde{S} \in sol(\mathbf{LP})$

ii) $(\widetilde{\pi}^b,(\widetilde{\pi}_i),(\widetilde{\pi}^i_{T_i})) \in \Pi$ where $\widetilde{\pi}^b = v(\widetilde{S}) - \sum_{k\in F(\widetilde{S})} \widetilde{p}^{\widetilde{S}}_k(\widetilde{S}_k)$, $\widetilde{\pi}_i = (\widetilde{p}^{\widetilde{S}}_i - c_i)(\widetilde{S}_i)$ and $\widetilde{\pi}^i_{T_i} = (\widetilde{p}^{\widetilde{S}}_i - c_i)(T_i)$ for all $i \in N, T_i \subseteq \Omega_i$.

Proof: See the Appendix.

By Corollary 3, $\widetilde{S} \subseteq \Omega$ is an equilibrium consumption set of G^{MB} if and only if $\widetilde{S} \in \arg\max_{S\subseteq\Omega}\{(v-c)(S)\}$. We show next the efficiency of equilibrium net prices. To this end, let us check first that the T-Core(G) coincides with the solutions of the dual linear problem,

Proposition 4 *Let v be a value function which defines the economy $G(n+1,v,c)$ and let $T \in \arg\max_{S\subseteq N}\{(v-c)(S)\}$. Then,*

i) if $(\pi^b,(\pi_i),(\pi^i_{T_i})) \in sol(\mathbf{DLP})$, then $(\pi^b,(\pi_i)) \in T$-Core(G),

ii) if $(\pi^b,(\pi_i)) \in T$-Core(G), then $(\pi^b,(\pi_i),(\pi^i_{T_i})) \in sol(\mathbf{DLP})$ where $\pi^i_{T_i} = \pi_i$ for all $i \in N, T_i \subseteq \Omega_i$.

Proof: First let us prove i). Without loss of generality suppose that $T = N$, then T-Core(G)=Core(G). Let $(\pi^b,(\pi_i),(\pi^i_{T_i})) \in sol(\mathbf{DLP})$ and $K \subseteq N$. Then, for all $T \subseteq \Omega$ such that $F(T) \subseteq K$,

$$\begin{aligned}
\pi^b + \sum_{i\in K}\pi_i &\geq \pi^b + \sum_{i\in F(T)}\pi_i \\
&\geq \pi^b + \sum_{T_i\in T}\pi^i_{T_i} \geq (v-c)(T)
\end{aligned}$$

where the first inequality holds given that $F(T) \subseteq K$ and the second and third inequalities are verified by constraints (5) and (4) respectively. Thus, $\pi^b + \sum_{i \in K} \pi_i \geq V(K)$. Moreover, by the fundamental duality theorem $V(N) = \pi^b + \sum_{i \in N} \pi_i$. Hence, $(\pi^b, (\pi_i)) \in Core(G)$.

To prove ii), let $(q^b, (q_i)) \in Core(G)$ and $S \in \arg\max_{T \subseteq \Omega}\{(v-c)(T)\}$, then trivially $q_i = 0$ for all $i \in N \backslash F(S)$. Define $(q^b, (q_i), (q^i_{T_i}))$ where $q^i_{T_i} = q_i$ for all $i \in N, T_i \subseteq \Omega_i$. Given $T \subseteq \Omega$, by condition ii) of the definition of Core(G),

$$q^b + \sum_{T_i \in T} q^i_{T_i} = q^b + \sum_{i \in F(T)} q_i \geq (v-c)(T)$$

and constraints (4) of **DLP** are satisfied. Moreover, by definition, $q_i \geq q^i_{T_i}$ for all $i \in N, T_i \subseteq \Omega_i$, thus also constraints (5) are satisfied. Finally, by condition i) of the definition of Core(G),

$$q^b + \sum_{i \in N} q_i = V(N) = (v-c)(S)$$

hence $(q^b, (q_i), (q^i_{T_i})) \in sol(\mathbf{DLP})$. ■

Then, by Propositions 3 and 4,

Corollary 4 *Let $(\widetilde{S}, \widetilde{p}^{\widetilde{S}})$ be an SPE^*-outcome. Then, the buyer surplus, $v(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} \widetilde{p}^{\widetilde{S}}_k(\widetilde{S}_k)$, and the firms' profit vector (or net price vector), $(\widetilde{p}^{\widetilde{S}} - c)(\widetilde{S})$, are a subset of $T-$Core(G), for any $T \subseteq N$.*

5 Monotonic social surplus functions

By Proposition 3, it is not difficult to show that if all socially efficient consumption sets of the game G^{MB} are such that $\widetilde{S} \subseteq \Omega_i$ for some firm $i \in N$, i.e., the buyer chooses only the products of a single firm, as in example 1, then $\widetilde{S}$ is sold as a bundle at price $p_i(\widetilde{S}) = v(\widetilde{S}) - \alpha$, where $\alpha = \max_{S \subseteq \Omega \backslash \Omega_i}\{(v-c)(S), 0\}$. Firm i obtains positive profits, the products of any other firm are offered at marginal cost prices and the buyer obtains a positive payoff equal to α. But, if there are two or more socially efficient pure consumption sets, then they are sold at marginal cost prices and the buyer obtains the entire surplus. On the other hand, when the equilibrium consumption set contains products of two o more firms, as in example 2, then although firms might offer their products as bundles for a special prices, the buyer selects a subset of products of each firm. Therefore, mixed bundling might also be an off-equilibrium pricing strategy, supporting equilibrium outcomes.

However, the precise form of equilibrium prices is difficult to obtain unless we know the specific value functions. In this section we characterize SPE^*-prices for monotonic social surplus functions. First, we need some definitions.

Definition 3 *$(v-c)$ is monotonic if and only if $(v-c)(S) \leq (v-c)(T)$ whenever $S \subseteq T \subseteq \Omega$.*

Then, monotonicity of $(v-c)$ implies that the social surplus increases for larger consumption sets.

The *social marginal contribution* of consumption set S, $c^*(S)$, for all $S \subseteq \Omega$, i.e., the increase in social surplus due to S, is

$$c^*(S) = (v-c)(\Omega) - (v-c)(\Omega \backslash S) = v(\Omega) - v(\Omega \backslash S) - c(S), \qquad \forall S \subseteq \Omega$$

If $(v-c)$ is monotonic, then, by Proposition 2, there exists an SPE^*-outcome with Ω as the equilibrium consumption set. Furthermore, if $(v-c)$ is strictly monotonic then Ω is the only equilibrium consumption set. Next Lemma shows that firm i's equilibrium profits have an upper bound.

Lemma 5 *If $(\Omega, p) \in SPE^*$-outcome set, then $p_i(\Omega_i) - c(\Omega_i) \leq c^*(\Omega_i)$ for all $i \in N$*

Proof: Let $i \in N$, by BC, $v(\Omega) - \sum_{k \in N} p_k(\Omega_k) \geq v(\Omega \backslash \Omega_i) - \sum_{k \in N \backslash i} p_k(\Omega_k)$ or equivalently,

$$v(\Omega) - v(\Omega \backslash \Omega_i) \geq \sum_{k \in N} p_k(\Omega_k) - \sum_{k \in N \backslash i} p_k(\Omega_k) = p_i(\Omega_i)$$

Thus, $c^*(\Omega_i) \geq p_i(\Omega_i) - c_i(\Omega_i)$. ■

Recall that $S_i = \Omega_i \cap S$, for all $i \in N$, then following Shapley (1962), we say that firms are *substitutes* if the social marginal contribution of consumption set S is bigger or equal to the sum of the social marginal contributions of firms in S,

$$c^*(S) \geq \sum_{S_i \in S} c^*(S_i) \qquad \forall S \subseteq \Omega \qquad \text{FS}$$

This property has been previously used in different setting by several authors as KC(1982) and BO(2002) among others. The former have employed it to justify that workers are better off by forming a union rather than by bargaining individually with management, whereas the latter to show that when buyers are substitutes, then the core has the lattice property with respect to buyers. Next Proposition says that equilibrium prices under monotonic social surplus functions satisfying FS are equal to marginal costs plus firms' social marginal contributions:

Proposition 5 *Let $(v-c)$ be a monotonic social surplus function and let firms be substitutes (FS holds). Then $(S^*, p^*) \in SPE^*$-outcome set, with $(v-c)(S^*) = (v-c)(\Omega)$ and $p_i^*(T_i) = c^*(\Omega_i) + c(T_i)$, for all $T_i \subseteq \Omega_i$, and for all $i \in N$. The converse is also true.*

If $(v-c)$ is strictly monotonic, then the unique SPE^*-outcome is $S^* = \Omega$

Proof: See the Appendix.

Each firm i sells S_i^* as a bundle at price

$$p_i^*(S_i^*) = c^*(\Omega_i) + c(S_i^*) \leq \sum_{w_i \in S_i^*} p_i^*(w_i)$$

obtaining its marginal contribution as its profits,

$$p_i^*(S_i^*) - c_i(S_i^*) = c^*(\Omega_i)$$

Moreover, the buyer surplus is positive, reflecting the market competition under FS:

$$\begin{aligned} cs &= v(S^*) - \sum_{S_i^* \in S^*} p_i^*(S_i^*) = (v-c)(S^*) - \sum_{S_i^* \in S^*} c_i^*(S_i^*) \\ &= c^*(S^*) - \sum_{S_i^* \in S^*} c_i^*(S_i^*) \geq 0 \end{aligned}$$

Finally,

Definition 4 (1) *$(v-c)$ is convex if and only if*

$$(v-c)(S+w) - (v-c)(S) \leq (v-c)(T+w) - (v-c)(T)$$

whenever $S \subseteq T \subseteq \Omega \backslash w$, and

(2) *$(v-c)$ is concave if the opposite holds, i.e.,*

$$(v-c)(S+w) - (v-c)(S) \geq (v-c)(T+w) - (v-c)(T)$$

5.1 Concave value functions

Concavity of $(v-c)$ reflects a kind of substitution among products or bundles of products so that there is market competition and the buyer will obtain some surplus. Two straightforward results are the following,

Lemma 6 *Let $(v-c)$ be a concave social surplus function and let $c^*(w) \geq 0$ for all $w \in \Omega$, then $(v-c)$ is monotonic.*

Proof: Let $w \neq w'$, $w, w' \in \Omega$. By concavity,

$$(v-c)(\Omega\backslash w) - (v-c)(\Omega\backslash\{w, w'\}) \geq (v-c)(\Omega) - (v-c)(\Omega\backslash w') = c^*(w') \geq 0$$

thus $(v-c)(\Omega) \geq (v-c)(\Omega\backslash w) \geq (v-c)(\Omega\backslash\{w, w'\})$. In general, by induction,

$$(v-c)(\Omega\backslash\{w_1, ..., w_l\}) \geq (v-c)(\Omega\backslash\{w_1, ..., w_l, w_{l+1}\})$$

i.e., $(v-c)(S) \leq (v-c)(S+w)$.

Let $S \subseteq T \subseteq \Omega$ and let $T\backslash S = \{w_1, ..., w_l\}$. We have that

$$(v-c)(S) \leq (v-c)(S \cup \{w_1\}) \leq ... \leq (v-c)(S \cup \{w_1, ..., w_{l-1}\}) \leq (v-c)(T).$$

■

Lemma 7 *Let $(v-c)$ be a concave social surplus function, then firms are substitutes, i.e. FS is satisfied.*

Proof: Let $S \subseteq \Omega$, $F(S) = \{i_1, ...i_l\}$. Then,

$$\begin{aligned} c^*(S) &= v(\Omega) - v(\Omega\backslash S) - c(S) = v(\Omega) - v(\Omega\backslash S_{i_1}) - c(S_{i_1}) \\ &\quad + \sum_{j=1}^{l-1} [v(\Omega\backslash\{S_{i_1}, ..., S_{i_j}\} - v(\Omega\backslash\{S_{i_1}, ..., S_{i_{j+1}}\} - c(S_{i_j})] \\ &\geq \sum_{j=1}^{l} v(\Omega) - v(\Omega\backslash S_{i_j}) - c(S_{i_j}) = \sum_{S_i \in S} c^*(S_i) \end{aligned}$$

■

Proposition 5 and Lemmas 6 and 7 make it possible to offer results for concave social surplus functions.

Corollary 5 *Let $(v-c)$ be concave and $c^*(w) \geq 0$, for all $w \in \Omega$. Then, firms are substitutes, and firms' equilibrium profits (or net prices) are equal to their social marginal contributions.*

5.2 Convex social surplus functions

Convexity of $(v-c)$ reflects complementarities among products or bundles of products and hence among firms. Therefore it induces only weak market competition so that firms can extract the entire buyer surplus. It is straightforward to prove that if $(v-c)$ is nonnegative and convex, then $(v-c)$ is monotonic.

Lemma 8 *Let $(v-c)$ be a convex social surplus function with $(v-c)(w) \geq 0$ for all $w \in \Omega$. Then $(v-c)$ is monotonic.*

The set of firms' equilibrium profits of convex social surplus functions is the convex hull of $n!$ vectors which are components $(\pi_i)_{i \in N}$ of the corner solutions of **RDLP**. Moreover, we show next that, if $(v-c)$ is monotonic and convex, then core$(v-c)$ is non-empty: in fact, core$(v-c)$ is the set of components $(\pi_i)_{i \in N}$ of the solutions of **RDLP**, thus coinciding with the set of firms' equilibrium profits. Let $P(\Pi) = \{(\pi_i) \in R^n_+ |$ there exists $(0, (\pi_i), (\pi^i_{T_i})) \in \Pi\}$.

Proposition 6 *Let $(v-c)$ be a convex value function, such that $(v-c)(w) \geq 0$ for all $w \in \Omega$. Then, the following two sets are the same.*

(i) core$(v-c)$

(ii) $P(\Pi) = \{\pi \in R^n_+ | (0, (\pi_i), (\pi^i_{T_i})) \in \Pi$ where $\pi^i_{T_i} = \pi_i$ for all $i \in N, T_i \subseteq \Omega_i\}$

and the buyer's surplus is zero.

Thus, if the social surplus function is convex, then the consumer's surplus is zero at any equilibrium outcome of the economy, so that firms extract the entire surplus.

6 The role of mixed bundling

Let us consider the scenario where firms are not allowed to used mixed bundling strategies. Thus, a strategy of firm j, $j \in N$, is a vector $p_j = \{p_j(w)\}_{w \in \Omega_j}$ where $p_j(w)$ is firm j's price of product w. Let $p_i(T_i)$ be the price of $T_i \subseteq \Omega_i$, then $p_i(T_i) = \sum_{w \in T_i} p_i(w)$, i.e., prices are linear and bundle T_i is offered for no special price. The buyer then select one consumption set as a function of the vector of prices $\{p_j(w)\}_{j \in N, w \in \Omega_j}$ that she already has observed. This defines a two-stage game[5] G^{LP}.

[5]A restricted version of this model is LU (2002).

Let us denoted a subgame perfect equilibrium outcome of G^{LP} as an $LSPE^*$-outcome. The properties that $LSPE^*$-outcomes shall satisfy are similar to BC1, FC1 and FC4. However, firms' possible deviations under linear prices are different from those in mixed bundling settings. Thus, condition FC2 is not of application here, given that the prices of two non-disjoint bundles are not independent anymore: if a firm increases the price of a product, it is simultaneously increasing the price of all the bundles containing it. As in the mixed bundling framework, $LSPE^*$-outcomes are associated to optimal solutions of some LP problems, denoted **LPL**. The **LPL** problem is similar to **LP** but we need to change constraints (3),

$$\sum_{S \ni T_i} z_S \leq y(T_i, i) \qquad \forall i \in N, \forall T_i \subseteq \Omega_i$$

by those associated to each firm i's product,

$$\sum_{S \ni w} z_S \leq \sum_{w \in T_i \subseteq \Omega_i} y(T_i, i) \qquad \forall i \in N, \forall w \subseteq \Omega_i$$

These new constraints tell us that $w \subseteq \Omega_i$ is sold by firm i if and only if the buyer has chosen a consumption set S with $w \in S$. Let **DLPL** be the associated dual problem and let $(\pi^b, (\pi_i), (\pi_w))$ be a generic solution of **DLPL**. Similar arguments to those of **LP** and **DLP** of section 3, ensures that **LPL** always has an integer optimal solution and hence (by the fundamental duality theorem) **DLPL** has an optimal solution with $\pi^b + \sum_{i \in N} \pi_i = VL(\Omega)$. Finally, the associated restricted dual problem, **RDLPL** is

$$\begin{array}{ll}
Max & \displaystyle\sum_{i \in N} \pi_i \\
s.t. & \pi^b + \displaystyle\sum_{w \in S} \pi_w \geq (v - c)(S) \qquad \forall S \subseteq \Omega \\
& \pi_i - \displaystyle\sum_{w \in T_i} \pi_w \geq 0 \qquad \forall i \in N, \ \forall T_i \subseteq \Omega_i \\
& \pi^b + \displaystyle\sum_{i \in N} \pi_i = VL(\Omega) \\
& \pi^b, \pi_i, \pi_w \geq 0
\end{array}$$

which will always have an optimal solution. It satisfies $sol(\mathbf{RDLPL}) \subseteq sol(\mathbf{DLPL})$. Given $(\pi^b, (\pi_i), (\pi_w)) \in sol(\mathbf{RDLPL})$, let $\pi^i_{T_i} = \sum_{w \in T_i} \pi_w$ for all $i \in N, T_i \subseteq \Omega_i$.

Notice that if **LP** and **DLP** problems of the Package Assignment model admit a degenerate solution in pure component prices, this solution is an $LSPE^*$-outcome. Let Π^L denote the set of optimal solutions of **DLPL** non Pareto-dominated on components (π_i) and recall that Π is the corresponding one for **DLP** problems. It is not difficult to show that if both frontiers have a non-empty intersection, i.e. if $(\pi^b, (\pi_i), (\pi_w)) \in \Pi^L \cap \Pi$, then $((\pi_i), (\pi_w))$ will define an $LSPE^*$-price and profits vectors (see AU, 2003b). However, when $(\pi^b, (\pi_i), (\pi_w)) \in \Pi^L \backslash \Pi$ different things may happen. Namely, although some solutions of the **LPL** and **DLPL** problems still define $LSPE^*$-outcomes, others do not characterize them anymore. This last case implying that either an $LSPE^*$-outcome does not exist or it is inefficient (see AU, 2003b, for the proofs of these results). To clarify the above discussion let us consider the following example in LU(2002), where existence of linear pricing equilibria is not always guaranteed.

Example: Let v be as in example 1. The following tables shows the $LSPE^*$-outcomes for linear (additive) prices.

Table 3: Equilibrium outcomes in linear prices

Equilibrium in Linear prices			
Region I $0 < \delta \leq 6$	Region II $6 < \delta < 12$	Region III $12 \leq \delta < 13$	Region IV $13 \leq \delta < 16$
$\widetilde{S} = \{a, d\}$ $p_1^a = 15 - \frac{2}{3}\delta$ $p_1^b = 14 - \frac{2}{3}\delta$ $p_2^c = \frac{2}{3}\delta$ $p_2^d = \frac{2}{3}\delta$ $cs = 0$ no efficient	No equilibrium exists	$\widetilde{S} = \{a, b\}$ $p_1^a = 2$ $p_1^b = 1$ $p_2^c = 0$ $p_2^d = 0$ $cs = 13$ efficient	$\widetilde{S} = \{a, b\}$ $p_1^a + p_1^b = 16 - \delta$ $p_1^a \leq 2, p_1^b \leq 1$ $p_2^c = 0$ $p_2^d = 0$ $cs = \delta$ efficient

In this example $LSPE^*$-outcomes can be inefficient (Region I) or could not exist (Region II). Notice first that for $13 \leq \delta < 16$, (region IV), $\{a, b\} \in sol(\mathbf{LPL})$ and $\pi^b = \delta, \pi_a = \min\{16 - \delta, 2\}, \pi_b = 16 - \delta - \pi_a, \pi_c = \pi_d = 0$ is a solution of **RDLPL** which also defines a solution of **RDLP**, i.e., $(\pi^b, (\pi_i), (\pi_w)) \in \Pi^L \cap \Pi$. Hence the outcome (S, p), where $S = \{a, b\}$ and $p(w) = \pi_w$ is an $LSPE^*$-outcome. Now, for $\delta < 13$ the unique solution of **RDLPL** is $\pi^b = 13, \pi_a = 2, \pi_b = 1, \pi_c = \pi_d = 0$ and it does not belong to Π anymore, i.e. $\Pi^L \cap \Pi = \emptyset$. However, for $12 \leq \delta < 13$ this solution still defines an $LSPE^*$-outcome. But, for $\delta < 12$, **LPL** and **RDLPL** solutions do not characterize $LSPE^*$-outcomes, this meaning that $LSPE^*$-outcomes could not exist or could be inefficient. For instance, let $\delta = 10$ and consider the optimal solution to **LPL** and **RDLPL** problems: outcome (S, p) where

$$\begin{aligned} S &= \{a, b\} \\ p_a &= 2, p_b = 1 \\ p_c &= p_d = 0. \end{aligned}$$

firm 1 obtains a profit of 3 and the buyer's surplus is 3. Firm 1 has incentives to change its prices so that $p_a = p_b = 5 - \varepsilon$, for ε small enough. Then, the buyer chooses bundle $\{a, d\}$, maximizing her surplus and firm 1 obtains a profit equal to $5 - \varepsilon$ bigger than 3.

Thus, there are important differences between games G^{MB} and G^{LP}: first, a subgame perfect equilibrium may not exists in linear prices (Region II); secondly, the equilibrium outcomes may be non-efficient (Region I). In the above example, when $0 < \delta \leq 6$ (Region I) the game G^{LP} has always a unique equilibrium: the buyer purchases the consumption set $\{a, d\}$, which is not socially efficient and the equilibrium prices are such that the firms extract the entire surplus, leaving the buyer with zero surplus, $v(a, d) = 15 = p_1^a + p_2^d$. However, if mixed bundling is allowed, then the unique equilibrium consumption set is $\{a, b\}$, the socially efficient. Firm 1 must sell its pure system as a bundle for the price $p_1^{ab} = 16 - \delta$, lower than the sum of the prices of its two products separately and firm 2 sets prices equal to zero. Thus, the buyer's surplus is δ. Observe that if $3 < \delta \leq 6$ then both firms are worse off in G^{MB} relatively to G^{LP}, but if $0 < \delta \leq 3$ then firm 1 is better off in G^{MB} and firm 2 is better off in G^{LP}. Now, let assume that $6 < \delta < 12$ (Region II), then no subgame perfect equilibria exists in G^{LP}, thus mixed bundling is needed to guarantee equilibrium's existence. This results are in contradiction with AL(1993). They found that firms will in equilibrium choose to precommit to linear prices better than allowing mixed bundling. This will be only the case in Region I when $3 < \delta \leq 6$. But, if firms have precommitted to linear prices in Region II, then the market would not achieve any equilibrium outcome meanwhile mixed bundling guarantees the socially efficient consumption set $\{a, b\}$, at price $p_1^{ab} = 16 - \delta$, and with a buyer's surplus of δ.

7 The Model with m buyers

Let us extend the above model to one with m buyers, denoted by $b \in B = \{1, ..., m\}$. Each buyer has a value function defined over sets of bundles of objects, $v_b(S)$, $S \subseteq 2^{\Omega}$. An *allocation* of objects to buyers is a vector $(S^1, ..., S^m)$ where $S^b \subseteq \Omega$, $S^b \cap S^{b'} = \emptyset$ for any $b, b' \in B$, $b \neq b'$. It is possible that for some b, $S^b = \emptyset$. Assume, for simplicity, that the unit costs of production are zero.

As in the previous model, each firm $i \in N$ chooses prices $p_i(T_i)$ for any set $T_i \subseteq \Omega_i$ and then, each buyer $b \in B$, after observing price vector $p = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, selects the consumption set $S^b \subseteq \Omega$ which maximizes her surplus, i.e., for all $b \in B$, and $S \subseteq \Omega$,

$$v_b(S^b) - \sum_{i \in F(S^b)} p_i(S_i^b) \geq v_b(S) - \sum_{i \in F(S)} p_i(S_i).$$

Let $(S^1, ..., S^m)$ be an allocation and denote by $S_i^b = S^b \cap \Omega_i$ the set of products sold by firm i to buyer b, hence $S = \cup_{b \in B} S^b$. Thus, firm i sells the set of products $S_i = \cup_{b \in B} S_i^b$, which can be arranged as a vector $(S_i^1, ..., S_i^m)$. Let Ψ_i be the set of such vectors associated to any possible allocation and $\psi_i \in \Psi_i$ any of its elements.

Given $b \in B, S \subseteq \Omega$, define $z(S, b)$ equal to 1 if buyer b chooses consumption set S, zero otherwise; for any $i \in N$ and $\psi_i \in \Psi_i$ define $y(\psi_i, i)$ equal to 1 if firm i sells its products to buyers according to ψ_i and zero otherwise. Consider the following integer problem **ILP'**.

$$\begin{aligned}
Max \quad & \sum_{b \in B} \sum_{S \subseteq \Omega} v_b(S) z(S, b) \\
s.t. \quad & \sum_{S \subseteq \Omega} z(S, b) \leq 1 \qquad \forall b \in B && (7)\\
& \sum_{\psi_i \in \Psi_i} y(\psi_i, i) \leq 1 \qquad \forall i \in N && (8)\\
& \sum_{b \in B} \sum_{\{S : S \cap \Omega_i = T\}} z(S, b) \leq \sum_{b \in B} \sum_{\{\psi_i : S_i^b = T\}} y(\psi_i, i), \;\; \forall i \in N, T \subseteq \Omega_i && (9)\\
& z(S, b), y(\psi_i, i) \in \{0, 1\}
\end{aligned}$$

where constraints (7) ensure that only one consumption set is chosen by each buyer, constraints (8) guarantee that each firm i sells every individual product to at most one buyer, and constraints (9) set that the bundle chosen by buyer b from firm i is the one sold by firm i to buyer b. Let **LP'** be the linear relaxation of **ILP'**, so that constraints $z(S,b), y(\psi_i, i) \in \{0,1\}$ change to $z(S,b), y(\psi_i, i) \geq 0$. Let $V_{ILP}(N,B)$ and $V_{LP}(N,B)$ denote the optimal value of **ILP'** and **LP'** respectively, thus $V_{LP}(N,B) \geq V_{ILP}(N,B)$. The **LP'** problem has associated the dual linear programming problem **DLP'**:

$$\begin{array}{lll}
Min & \sum_{b \in B} \pi^b + \sum_{i \in N} \pi^i & \\
s.t. & \pi^b + \sum_{i \in N} \pi^i_{S \cap \Omega_i} \geq v_b(S) & \forall b \in B, S \subseteq \Omega \\
& \pi^i - \sum_{b \in B} \pi^i_{S^b_i} \geq 0 & \forall i \in N, \forall \psi_i = (S^1_i, ..., S^m_i) \in \Psi_i \\
& \pi^b, \pi^i, \pi^i_T \geq 0 &
\end{array}$$

where π^i can be interpreted as firm i's net profit, $i \in N$; π^b is buyer b's surplus; and π^i_T is the price that firm i sets to bundle T, $i \in N, T \subseteq \Omega_i$. The feasible region of **LP'** is nonempty, which implies that **DLP'** also has an optimal solution, say $V_{DLP}(N,B)$. Among the optimal solutions of **DLP'** let us consider those whose coordinates (π^i), for all $i \in N$, are not Pareto-dominated by any other optimal solution. A way to obtain some of these solutions is to consider the following restricted dual problem **RDLP'**,

$$\begin{array}{lll}
Max & \sum_{i \in N} \pi^i & \\
s.t. & \pi^b + \sum_{i \in N} \pi^i_{S \cap \Omega_i} \geq v_b(S) & \forall b \in B, S \subseteq \Omega \\
& \pi^i - \sum_{b \in B} \pi^i_{S^b_i} \geq 0 & \forall i \in N, \forall \psi_i = (S^1_i, ..., S^m_i) \in \Psi_i \\
& \sum_{b \in B} \pi^b + \sum_{i \in N} \pi^i = V_{DLP}(N,B) & \\
& \pi^b, \pi^i, \pi^i_T \geq 0 &
\end{array}$$

By the duality theorem of linear programming we have that

$$V_{DRLP}(N,B) = V_{DLP}(N,B) = V_{LP}(N,B) \geq V_{ILP}(N,B).$$

As an extension of Proposition 2 it can be proven that if $V_{LP}(N,B) = V_{ILP}(N,B)$, then an optimal solution of **LP'** together with a non Pareto-dominated solution of **DLP'** in coordinates (π^i), for all $i \in N$, yield an

SPE^*-outcome of the m- buyers' model. The equilibrium price vectors are mixed bundling prices. Thus, the existence of SPE^*-outcomes depends on the existence of an integer optimal solution of **LP'**.

Trivially, if all buyers have additive value functions ($v_b(S) + v_b(T) = v_b(S \cup T)$ for all $b \in B$ and $S, T \subseteq \Omega$) then, equilibrium prices will also be linear, with $p_i(w) = \max_b\{v_b(w)\}$, for all firm $i \in N, w \in \Omega_i$ and existence of (degenerate) SPE^*-outcomes is always guaranteed. The next sub-sections analyze conditions on buyers' value functions which ensure existence of SPE^*-outcomes.

7.1 Homogeneous buyers

Let us first analyze the case in which all value functions are the same. We need conditions to ensure that $\sum_{b\in B}\sum_{S\subseteq\Omega} v_b(S)z(S,b)$ is maximized at an integer feasible allocation. Let $(\lambda_S)_{S\subseteq\Omega}$, be a balanced vector if $\lambda_S \geq 0$ for all $S \subseteq \Omega$ and for all $w \in \Omega, \sum_{S\ni w}\lambda_S = 1$. A value function $v(.)$ defined on subsets $S \subseteq \Omega$ is balanced if for any balanced vector $(\lambda_S)_{S\subseteq\Omega}$ we have $\sum_S \lambda_S v(S) \leq v(\Omega)$.

Result 1: If all agents have the same value function $v(.)$, and if $v(.)$ is balanced, then $V_{LP}(N,B) = V_{ILP}(N,B)$ and there exists an SPE^*-outcome (as solution of the primal and dual linear problems).

For homogeneous balanced value functions,

$$\sum_{b\in B}\sum_{S\subseteq\Omega} v(S)z(S,b) \leq v(\Omega)$$

thus **LP'** has an optimal integer solution determined by, say, allocation $(\Omega, \emptyset, ..., \emptyset)$. More precisely, $(\widetilde{z}(S,b), \widetilde{y}(\psi_i,i))$ is an optimal solution of **LP'**, where $\widetilde{z}(\Omega,1) = 1$ and $\widetilde{z}(S,b) = 0$ if $S \neq \Omega$ or $b \neq 1$ and $\widetilde{y}(\psi_i,i) = 1$ if $\psi_i = (\Omega_i, \emptyset, ..., \emptyset)$ for all $i \in N$, and zero otherwise.

Proof: Let $((z(S,b),(y(\psi_i,i))$ be a feasible solution of **LP'**, thus for all $w \in \Omega$, $\sum_{S\ni w}\sum_{b\in B} z(S,b) \leq 1$. Define $\lambda_w = \sum_{b\in B} z(w,b) + (1 - \sum_{S\ni w}\sum_{b\in B} z(S,b))$. For $S \subseteq \Omega$, such that $|S| \geq 2$ define $\lambda_S = \sum_{b\in B} z(S,b)$. It can be checked that $(\lambda_S)_{S\subseteq\Omega}$ is a balanced vector and $\lambda_S \geq \sum_{b\in B} z(S,b)$. Thus,

$$\sum_{b\in B}\sum_{S\subseteq\Omega} v(S)z(S,b) = \sum_{S\subseteq\Omega}\sum_{b\in B} z(S,b)v(S) \leq \sum_{S\subseteq\Omega}\lambda_S v(S) \leq v(\Omega)$$

where the first inequality is given by $\lambda_S \geq \sum_{b\in B} z(S,b)$ and the second one from balancedness. ■

Moreover, given that a convex value function is balanced we conclude that if all agents have the same convex value function, then the set of SPE^*-outcomes is non empty.

7.2 Heterogeneous buyers

The existence of SPE^*-outcomes can be easily extended to two types of buyers, provided they have strictly convex value functions i.e., there are two types of buyers such that agents type $j = 1, 2$ have the same convex value function.

To see this, let B^1 be the set of type 1 buyers and B^2 be the set of type 2. It is not difficult to show that at any optimal solution of **DRLP'** the buyers' surplus is the same for all of them and equal to zero (the proof runs similar to the single buyer model).

Case 1: In the optimal solution of **ILP'**, there exists $b \in B^1$ such that $V_{ILP}(N, B) = v_b(\Omega)$. Relabeling the buyers if necessary, $b = 1$. Consider the allocation, $\widetilde{\psi} = (\Omega, \emptyset, ..., \emptyset)$ and price vector $\widetilde{p} = (\widetilde{p}_1, ..., \widetilde{p}_n)$ such that,

$$\sum_{i \in N} p_i(\Omega_i) = v_1(\Omega)$$
$$p_i(\Omega_i) \leq v_1(\Omega) - v_1(\Omega \backslash \Omega_i) \text{ for all } i \in N$$
$$p_i(T) = p_i(\Omega_i) \text{ for all } i \in N, T \subseteq \Omega_i$$

where conditions 1 and 2 are simultaneously verified by convexity of v_1. It can be checked that this allocation and price vector define an equilibrium outcome.

Case 2: In the optimal solution of **ILP'**, there exists $b \in B^1$ and $b' \in B^2$ such that $V_{ILP}(N, B) = v_b(S) + v_{b'}(T)$ for some $S, T \subseteq \Omega$, with $S \cap T = \emptyset$. Relabeling the buyers if necessary, $b = 1, b' = 2$. Also notice that by convexity of the value functions, cases 1 and 2 are the unique integer optimal solutions .

Consider the following allocation, $\widetilde{\psi} = (S, T, \emptyset, ..., \emptyset)$ and the price vector $\widetilde{p} = (\widetilde{p}_1, ..., \widetilde{p}_n)$ such that,

$$\sum_{S_i \in S} p_i(S_i) = v_1(S)$$
$$\sum_{T_i \in T} p_i(T_i) = v_2(T)$$
$$p_i(K) = p_i(S_i) + p_i(T_i) \text{ for all } i \in N, K \subseteq \Omega_i$$

and $p_i(T_i)$, $p_i(S_i)$ bigger than some amount characterized by the constraints of **DLP'**. It can be checked that this allocation and price vector defines an equilibrium outcome.

The extension of this result to more than two types of buyers remains an open question. For instance, in the three types case the existence of Walrasian equilibria with linear prices is not guaranteed as shown in the following example with one firm, and three types of buyers (see, BM, 1996):

Example in BM (1996)

S	1	2	3	$1,2$	$1,3$	$2,3$	$1,2,3$
v_1	1	1	1	30	3	3	40
$v2$	1	1	1	3	30	3	40
$v3$	1	1	1	3	3	30	40

However, solving the linear programming problem and it restricted dual we find the following SPE^*-outcome,

$$\begin{aligned} \psi &= (\{1,2,3\},\emptyset,\emptyset) \\ p(S) &= 40 \text{ for all } S \subseteq \Omega \end{aligned}$$

Thus, mixed bundling prices guarantee the existence of efficient equilibrium outcomes. Our intuition suggests that under strict convexity of the buyers' value functions there always exist SPE^*-outcomes in mixed bundling prices, independently of the number of types of the buyers. Nevertheless, something more precise can be said if we consider a single firm model.

Result 2: Let $N = 1$, a single seller, and m buyers with strictly convex value functions. Then the set of SPE^*-outcomes is non empty.

Proof: Let $(\widetilde{z}, \widetilde{y})$ be an optimal solution of **ILP'** which defines the allocation $(\widetilde{S}^1, ..., \widetilde{S}^m)$. By convexity, $v_{b'}(\widetilde{S}^b) \leq v_b(\widetilde{S}^b)$, for every $\widetilde{S}^b \neq \emptyset$ and for all $b, b' \in B, b \neq b'$. Suppose that there exists $b, b' \in B$, and $v_{b'}(\widetilde{S}^b) > v_b(\widetilde{S}^b)$ then,

$$v_b(\widetilde{S}^b) + v_{b'}(\widetilde{S}^{b'}) < v_{b'}(\widetilde{S}^b) + v_{b'}(\widetilde{S}^{b'}) \leq v_{b'}(\widetilde{S}^b \cup \widetilde{S}^{b'})$$

which contradict the optimality of $(\widetilde{S}^1, ..., \widetilde{S}^m)$.

Now define $\widetilde{\pi}^b = 0$ for all $b \in B$, $\widetilde{\pi}_1 = \sum_{b\in B} v_b(\widetilde{S}_1^b)$ and $\widetilde{\pi}_T^1 = \max_b v_b(T)$. It can be checked that $((\widetilde{\pi}^b), \widetilde{\pi}_1, (\widetilde{\pi}_T^1))$ is a feasible solution of **DLP'**, thus $V_{DLP}(1,b) \leq \sum_{b\in B} v_b(\widetilde{S}_1^b)$

Then,

$$V_{ILP}(1,B) \leq V_{LP}(1,b) = V_{DLP}(1,b) \leq \sum_{b\in B} v_b(\widetilde{S}_1^b)$$

but $\sum_{b\in B} v_b(\widetilde{S}_1^b) \leq V_{ILP}(1,B)$ and we conclude that $V_{ILP}(1,B) = V_{LP}(1,b)$, $(\widetilde{S}^1,...,\widetilde{S}^m)$ is an optimal allocation of both **ILP'** and **LP'**, and jointly with the vector of prices given by $(\widetilde{\pi}_T^1)$ defines an equilibrium outcome. ■

Finally, consider the case in which all buyers are characterized by a value function $v_b, b \in B$ verifying KC(1982)'s gross substitution condition (gs). Given a buyer $b \in B$, with value function v_b and a price vector p, let $D_b(p)$ be

$$D_b(p) = \{S \subseteq \Omega | v_b(S) - \sum_{S_i \in S} p_i(S_i) \geq v_b(T) - \sum_{T_i \in T} p_i(T_i) \text{ for all } T \subseteq \Omega\}$$

Now, define (gs) as follows,

Given $b \in B$, v_b satisfies gross substitution if for any two price vectors p and q such that $p_i(T) \leq q_i(T)$ for all $i \in N, T \subseteq \Omega_i$ and any $S \in D_b(p)$, there exists $T \in D_b(q)$ such that $T_i = S_i$ if $p_i(S_i) = q_i(S_i)$.

The results in GS(1999) and BVSV(2002) can be extended to cover SPE^*-outcomes under mixed bundling pricing. Let $r_b(S,p) = \min_{T\in D_b(p)} |S \cap T|$ be the dual rank function of a matroid. Then by the matroid partition theorem if for all $T \subseteq \Omega$, $\sum_{b\in B} r_b(T,p) \leq |T|$, then there exists a partition of Ω so that every buyer receives at most one element of $D_b(p)$. In other case, choose $T' \subseteq \Omega$, such that $\sum_{b\in B} r_b(T,p) > |T|$ and is the one verifying that property with minimal cardinality. Now, following a modification of the algorithm proposed by GS(2000), each firm i increases the price of its bundles $S_i \subseteq \Omega_i$ such that $S_i \cap T_i \neq \emptyset$ by $\epsilon > 0$. After a finite number of rounds we obtain a price vector and an allocation which defines an optimal solution of both **LP'** and **ILP'**.

We conclude by noting that existence of SPE^*-outcomes for the m-buyers' model under general value functions is hardly guaranteed. However, our intuition suggests that if we allow mixed bundling prices to be also non-anonymous (i.e. buyer's dependent), some partial results could be given (see BO, 2002, for the Walrasian Equilibrium set up). This is left for future research.

8 Appendix.

Proof of Lemma 3*:* If $i \notin F(\widetilde{S})$, then constraint $\sum_{T_i \subseteq \Omega_i} y(T,i) \leq 1$ is satisfied with strict inequality and by the complementary slackness condition $\pi_i = 0$. Now, by constraints (5) in **DLP**, $0 = \pi_i \geq \pi^i_{T_i} \geq 0$ and ii) is satisfied.

Now we prove i): if $i \in F(\widetilde{S})$, again by constraints (5), $\pi_i \geq \pi^i_{T_i}$ for all $T_i \subseteq \Omega_i$. Given that $z_{\widetilde{S}} = 1$, then by the complementary slackness condition, $\pi^b + \sum_{\widetilde{S}_i \in \widetilde{S}} \pi^i_{\widetilde{S}_i} = v(\widetilde{S})$. Thus,

$$\pi^b = v(\widetilde{S}) - \sum_{i \in N} \pi_i = v(\widetilde{S}) - \sum_{\widetilde{S}_i \in \widetilde{S}} \pi^i_{\widetilde{S}_i}$$

which implies that $\sum_{i \in N} \pi_i = \sum_{\widetilde{S}_i \in \widetilde{S}} \pi^i_{\widetilde{S}_i} = \sum_{i \in F(\widetilde{S})} \pi_i$. But this, in turn, implies that for all $i \in F(\widetilde{S})$, $\pi_i = \pi^i_{\widetilde{S}_i}$ given that by (5) $\pi_i \geq \pi^i_{\widetilde{S}_i}$. ■

Proof of Proposition 2: W.l.o.g., we assume, for simplicity, that marginal costs are zero. We prove first that $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set under partition $\mu = \{N\}$ (μ-**DLP**=**RDLP**) and $\widetilde{p}_i(T_i) = \widetilde{\pi}^i_{T_i}$. To this end, we check that conditions BC, FC1 and FC4 are satisfied.

Step 1: Condition BC

Given $S \subseteq \Omega$, by Lemma 3 and constraints (4) in **DLP** (or **RDLP**),

$$\begin{aligned} v(S) &\leq \widetilde{\pi}^b + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} = v(\widetilde{S}) - \sum_{\widetilde{S}_i \in \widetilde{S}} \widetilde{\pi}^i_{\widetilde{S}_i} + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} = \\ &= v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) + \sum_{i \in F(S)} \widetilde{p}_i(S_i) \end{aligned}$$

thus $v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}) \geq v(S) - \sum_{i \in F(S)} \widetilde{p}_i(S)$.

Step 2: Condition FC1

If $v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) = 0$, then it is trivially verified for $S^j = \emptyset$.

Now suppose that $\widetilde{\pi}^b = v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) > 0$ and that there exists $j \in F(\widetilde{S})$ such that for all $S' \subseteq \Omega \backslash \Omega_j$,

$$\widetilde{\pi}^b = v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) > v(S') - \sum_{i \in F(S')} \widetilde{p}_i(S'_i)$$

Let $\epsilon = \frac{1}{2} \min_{S \subseteq \Omega \backslash \Omega_j} \{v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) - (v(S) - \sum_{i \in F(S)} \widetilde{p}_i(S_i))\} > 0$.

Define,

$$\begin{aligned}
\widehat{\pi}^b &= \widetilde{\pi}^b - \epsilon \\
\widehat{\pi}_i &= \begin{cases} \widetilde{\pi}_i + \epsilon & if \quad i = j \\ \widetilde{\pi}_i & if \quad i \neq j \end{cases} \\
\widehat{\pi}^i_{T_i} &= \begin{cases} \widetilde{\pi}^i_{T_i} + \epsilon & if \quad i = j, T_i = \widetilde{S}_i \\ \widetilde{\pi}^i_{T_i} & otherwise \end{cases}
\end{aligned}$$

Let us show that $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i}))$ satisfies all **RDLP** constraints. Given $S \subseteq \Omega$, if $S_j = \widetilde{S}_j$ then

$$\widehat{\pi}^b + \sum_{S_i \in S} \widehat{\pi}^i_{S_i} = \widetilde{\pi}^b - \epsilon + \sum_{S_i \in S \setminus S_j} \widetilde{\pi}^i_{S_i} + \widetilde{\pi}^j_{S_j} + \epsilon = \widetilde{\pi}^b + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} \geq v(S)$$

otherwise,

$$\begin{aligned}
\widehat{\pi}^b + \sum_{S_i \in S} \widehat{\pi}^i_{S_i} &= \widetilde{\pi}^b - \epsilon + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} \geq \\
&\geq \widetilde{\pi}^b - \frac{1}{2}(v(\widetilde{S}) - \sum_{i \in N} \widetilde{p}_i(\widetilde{S}) - v(S) + \sum_{i \in N} \widetilde{p}_i(S)) + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} = \\
&= \widetilde{\pi}^b - \frac{1}{2}(v(\widetilde{S}) - \sum_{S_i \in S} \widetilde{\pi}^i_{\widetilde{S}_i} - v(S) + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i}) + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} = \\
&= \widetilde{\pi}^b - \frac{1}{2}(\widetilde{\pi}^b - v(S) + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i}) + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} = \\
&= \frac{1}{2}(v(S) + \widetilde{\pi}^b + \sum_{S_i \in S} \widetilde{\pi}^i_{S_i}) \geq \frac{1}{2}(v(S) + v(S)) = v(S)
\end{aligned}$$

Thus, constraints (4) are satisfied. To prove that so do constraints (5), consider different cases. For all $i \neq j$, $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \widetilde{\pi}_i - \widetilde{\pi}^i_{T_i} \geq 0$. If $i = j$ but $T_i \neq \widetilde{S}_i$ then $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \widetilde{\pi}_i + \epsilon - \widetilde{\pi}^i_{T_i} \geq \widetilde{\pi}_i - \widetilde{\pi}^i_{T_i} \geq 0$; if $i = j$ and $T_i = \widetilde{S}_i$ then $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \widetilde{\pi}_i + \epsilon - (\widetilde{\pi}^i_{T_i} + \epsilon) = \widetilde{\pi}_i - \widetilde{\pi}^i_{T_i} \geq 0$.

Finally, $\widehat{\pi}^b + \sum_{i \in N} \widehat{\pi}_i = \widetilde{\pi}^b - \epsilon + \sum_{i \in N \setminus j} \widetilde{\pi}_i + \widetilde{\pi}_j + \epsilon = \widetilde{\pi}^b + \sum_{i \in N} \widetilde{\pi}_i = V(\Omega)$. Then $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i}))$ verifies all **RDLP** constraints and $\sum_{i \in N} \widehat{\pi}_i = \sum_{i \in N} \widetilde{\pi}_i + \epsilon > \sum_{i \in N} \widetilde{\pi}_i$ which contradicts the optimality of $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i}))$.

Step 3: Condition FC4

Let $A \subseteq N \setminus F(\widetilde{S})$, $B \subseteq F(\widetilde{S})$. First, recall, by Lemma 3, that $\widetilde{\pi}_i = \widetilde{\pi}^i_{T_i} = 0$ for all $i \in A, T_i \subseteq \Omega_i$. Second, by the dual constraints (5) and Lemma 3 that $\widetilde{\pi}_i = \widetilde{\pi}^i_{\widetilde{S}_i} \geq \widetilde{\pi}^i_{T_i}$ for all $i \in B, T_i \subseteq \Omega_i$.

Then, given $S \subseteq \Omega$, where $(A \cup B) \subseteq F(S)$,

$$\begin{aligned}
v(\widetilde{S}) - \sum_{i \in F(\widetilde{S})} \widetilde{p}_i(\widetilde{S}_i) &= v(\widetilde{S}) - \sum_{\widetilde{S}_i \in \widetilde{S}} \widetilde{\pi}^i_{\widetilde{S}_i} \geq v(S) - \sum_{S_i \in S} \widetilde{\pi}^i_{S_i} \\
&= v(S) - \sum_{\substack{S_i \in S \\ i \notin (A \cup B)}} \widetilde{\pi}^i_{S_i} - \sum_{\substack{S_i \in S \\ i \in B}} \widetilde{\pi}^i_{S_i} \\
&\geq v(S) - \sum_{i \in F(S) \setminus (A \cup B)} \widetilde{p}_i(S_i) - \sum_{i \in B} \widetilde{p}_i(\widetilde{S}_i)
\end{aligned}$$

and FC4 is verified.

Thus, $(\widetilde{S}, \widetilde{p})$ verifies conditions BC, FC1 and FC4, and hence $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set when $\mu = \{N\}$.

The proof for any other partition $\mu \neq \{N\}$ is not remarkably different from case $\mu = \{N\}$. Finally the proposition holds if we consider any price vector such that $\widetilde{\pi}^i_{T_i} \leq \widetilde{p}_i(T_i) \leq \widetilde{\pi}_i$ for all $i \in N, T_i \subseteq \Omega_i$ given that if $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mu{-}\mathbf{DLP})$, then $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\pi^i_{T_i})) \in sol(\mu{-}\mathbf{DLP})$, where $\widetilde{\pi}^i_{T_i} \leq \pi^i_{T_i} \leq \widetilde{\pi}_i$ for all $i \in N, T_i \subseteq \Omega_i$. ■

To prove Corollary 1 we need the following lemma which establishes that given $(S, p) \in SPE^*$-outcome set then, the net price vector $(p^S - c(S))$ is a solution of **DLP**, where vector p^S is defined from p as in page 11.

Lemma 9 *If $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set, then*

i) $\widetilde{S} \in sol(\mathbf{LP})$ and

ii) $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mathbf{DLP})$ where $\widetilde{\pi}^b = v(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} \widetilde{p}_k(\widetilde{S}_k)$ and $\widetilde{\pi}_i = \widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S})$ for all $i \in F(\widetilde{S})$, $\widetilde{\pi}_i = 0$ for all $i \notin F(\widetilde{S})$ and $\widetilde{\pi}^i_{T_i} =$

$$\widetilde{\pi}^i_{T_i} = \begin{cases} \widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) & \text{if } \; i \in F(\widetilde{S}), \widetilde{p}_i(T_i) - c_i(T_i) \geq \widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) \\ \widetilde{p}_i(T_i) - c_i(T_i) & \text{if } \; i \in F(\widetilde{S}), \widetilde{p}_i(T_i) - c_i(T_i) < \widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) \\ 0 & \text{if } \; i \notin F(\widetilde{S}) \end{cases}$$

for all $i \in N, T_i \subseteq \Omega_i$.

Proof of lemma 9: $\widetilde{S} \in sol(\mathbf{LP})$ by Corollary 3. Now, let us prove that $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mathbf{DLP})$:

Step 1: Constraints (4).

Given $S \subseteq \Omega$, let $A = F(S) \setminus F(\widetilde{S})$ and $B = \{i \in F(\widetilde{S}) | (\widetilde{p}_i - c_i)(\widetilde{S}_i) \leq (\widetilde{p}_i - c_i)(S_i)\} \cap F(S)$. Thus, $\widetilde{\pi}^k_{S_k} = 0$ for all $k \in A$ and $\widetilde{\pi}^k_{S_k} = \widetilde{\pi}^k_{\widetilde{S}_k}$ for all

$k \in B$. By FC3 it is verified that,

$$\begin{aligned}
\widetilde{\pi}^b &= v(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} \widetilde{p}_k(\widetilde{S}_k) = (v-c)(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} \widetilde{\pi}^k_{\widetilde{S}_k} \\
&\geq (v-c)(S) - \sum_{k \in F(S) \setminus (A \cup B)} (\widetilde{p}_k - c_k)(S_k) - \sum_{k \in B} (\widetilde{p}_k - c_k)(\widetilde{S}_k) \\
&= (v-c)(S) - \sum_{k \in F(S) \setminus (A \cup B)} \widetilde{\pi}^k_{S_k} - \sum_{k \in B} \widetilde{\pi}^k_{\widetilde{S}_k} \\
&= (v-c)(S) - \sum_{k \in F(S) \setminus (A \cup B)} \widetilde{\pi}^k_{S_k} - \sum_{k \in B} \widetilde{\pi}^k_{S_k} - \sum_{k \in A} \widetilde{\pi}^k_{S_k} \\
&= (v-c)(S) - \sum_{k \in F(S)} \widetilde{\pi}^k_{S_k}
\end{aligned}$$

Hence, $\widetilde{\pi}^b + \sum_{S_k \in S} \widetilde{\pi}^k_{S_k} \geq (v-c)(S)$.

Step 2: Constraints (5).

By definition $\widetilde{\pi}_i - \widetilde{\pi}^i_{T_i} \geq 0$.

We have seen that $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i}))$ is a feasible solution. In the next step we will see that is also optimal

Step 3: Optimality.

Given that,

$$\widetilde{\pi}^b + \sum_{k \in N} \widetilde{\pi}_k = (v-c)(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} (\widetilde{p}_k - c_k)(\widetilde{S}_k) + \sum_{k \in F(\widetilde{S})} (\widetilde{p}_k - c_k(\widetilde{S}_k) = (v-c)(\widetilde{S}) = V(\Omega)$$

thus, $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_{T_i})) \in sol(\mathbf{DLP})$. ■

Proof of corollary 1: By Proposition 2 when $\mu = \{N\}$, then $(\widetilde{S}, \widetilde{p}) \in SPE^*$-outcome set. Now, let us consider any other (S, p) in set SPE^*. By lemma 9, $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{DLP})$ where $(\pi^b, (\pi_i), (\pi^i_{T_i}))$ is defined as above. Moreover, by Corollary 3, $(v-c)(\widetilde{S}) = (v-c)(S)$.

Thus, given that $(\widetilde{\pi}^b, (\widetilde{\pi}_i), (\widetilde{\pi}^i_T)) \in sol(\mathbf{RDLP})$ it must be verified that $\sum_{k \in N} \widetilde{\pi}_k \geq \sum_{k \in N} \pi_k$. Hence,

$$\begin{aligned}
v(\widetilde{S}) - \sum_{k \in F(\widetilde{S})} \widetilde{p}_k(\widetilde{S}_k) &= \widetilde{\pi}^b = (v-c)(\widetilde{S}) - \sum_{k \in N} \widetilde{\pi}_k \\
&\leq (v-c)(S) - \sum_{k \in N} \pi_k \\
&= \pi^b = v(S) - \sum_{k \in F(S)} \widetilde{p}_k(S_k)
\end{aligned}$$

and the consumer surplus associated to $(\widetilde{S},\widetilde{p})$ is the lowest among all the SPE-outcomes. ■

Proof of Proposition 3*:* Let $(\widetilde{S},\widetilde{p}^{\widetilde{S}})$ be in set SPE^{*}. By Corollary 3, $\widetilde{S}\in sol(\mathbf{LP})$ and by Lemma 9, $(\widetilde{\pi}^{b},(\widetilde{\pi}_i),(\widetilde{\pi}^{i}_{T_i}))\in sol(\mathbf{DLP})$.

Moreover, by Lemma 4, $(\widetilde{p}-c)$ is not Pareto dominated by any other SPE^{*}-net price vector and nor is $(\widetilde{\pi}_i)$.

Now, suppose that $\widetilde{S}\in sol(\mathbf{LP})$, $(\widetilde{\pi}^{b},(\widetilde{\pi}_i),(\widetilde{\pi}^{i}_{T_i}))\in\Pi$. Let $\widetilde{p}^{\widetilde{S}}$ be such that $\widetilde{p}^{\widetilde{S}}_i(S_i)=\widetilde{\pi}^{i}_{S_i}+c_i(S_i)$ for all $i\in N, S_i\subseteq\Omega_i$

Step 1: Condition BC.

Given $S\subseteq\Omega$, by constraint (4) $\widetilde{\pi}^{b}+\sum_{S_i\in S}\widetilde{\pi}^{i}_{S_i}\geq(v-c)(S)$, but $\widetilde{\pi}^{b}=(v-c)(\widetilde{S})-\sum_{\widetilde{S}_i\in\widetilde{S}}\widetilde{\pi}^{i}_{\widetilde{S}_i}$ thus,

$$(v-c)(\widetilde{S})-\sum_{\widetilde{S}_i\in\widetilde{S}}\widetilde{\pi}^{i}_{\widetilde{S}_i}\geq(v-c)(S)-\sum_{S_i\in S}\widetilde{\pi}^{i}_{S_i}$$

and hence $v(\widetilde{S})-\sum_{k\in F(\widetilde{S})}\widetilde{p}^{\widetilde{S}}_k(\widetilde{S}_k)\geq v(S)-\sum_{k\in F(S)}\widetilde{p}^{\widetilde{S}}_k(S_k)$.

Step 2: Condition FC1.

If $\widetilde{\pi}^{b}=0$ then FC1 holds for $S^{j}=\emptyset$. If $\widetilde{\pi}^{b}>0$ then suppose there exits $j\in F(\widetilde{S})$ such that for all $S\subseteq\Omega\backslash\Omega_j$ it is verified that

$$v(\widetilde{S})-\sum_{k\in F(\widetilde{S})}\widetilde{p}^{\widetilde{S}}_k(\widetilde{S}_k)>v(S)-\sum_{k\in F(S)}\widetilde{p}^{\widetilde{S}}_k(S_k)$$

We can define $(\pi^{b},(\pi_i),(\pi^{i}_{T_i}))$ as in step 2 of the proof of the Proposition 2 which verifies all the constraint in **DLP** and Pareto dominates $(\widetilde{\pi}^{b},(\widetilde{\pi}_i),(\widetilde{\pi}^{i}_{T_i}))$, contradiction.

Step 3: Condition FC4, reproduce the same reasoning that step 3 of the proof of Proposition 2.

Hence $(\widetilde{S},\widetilde{p}^{\widetilde{S}})\in SPE^{*}$-outcome set. ■

Proof of Proposition 5: W.l.o.g., we prove the proposition assuming for simplicity that marginal costs are zero. Given that v is monotonic $v(\Omega)\geq v(S)$. Moreover,

$$p^{*}_i(T_i)\geq v(\Omega\backslash(\Omega_i\backslash T))-v(\Omega\backslash\Omega_i)$$

Thus, prices are all positive and $p^{*}_i(\Omega_i)=v(\Omega)-v(\Omega\backslash\Omega_i)$.

First we prove that $(\Omega,p^{*})\in SPE^{*}$-outcome set, by showing that $\Omega\in sol(\mathbf{LP})$ and $(\pi^{b},(\pi_i),(\pi^{i}_{T_i}))\in sol(\mathbf{RDLP})$ where $\pi^{i}_{T_i}=p^{*}_i(T_i)$, $\pi_i=p^{*}_i(\Omega_i)$ and $\pi^{b}=v(\Omega)-\sum_{i\in N}p^{*}_i(\Omega_i)$.

Step 1: $\pi^b \geq 0$. By (FS) $c^*(\Omega) = v(\Omega) \geq \sum_{i\in N} c^*(\Omega_i) = \sum_{i\in N} p_i^*(\Omega_i)$, thus $\pi^b \geq 0$.

Step 2: Constraints (4) in **RDLP** are verified.

Given $S \subseteq \Omega$, by (FS), $c^*(\Omega \backslash S) = v(\Omega) - v(S) \geq \sum_{i\in N} c^*(\Omega_i \backslash S)$, but

$$\begin{aligned}
v(\Omega) - v(S) &\geq \sum_{i\in N} c^*(\Omega_i \backslash S) = \\
&= \sum_{i\in N} [v(\Omega) - v(\Omega \backslash (\Omega_i \backslash S))] = \\
&= \sum_{i\in N} [v(\Omega) - v(\Omega \backslash \Omega_i) + v(\Omega \backslash \Omega_i) - v(\Omega \backslash (\Omega_i \backslash S))] \geq \\
&= \sum_{i\in N} [\pi^i_{\Omega_i} - \pi^i_{S_i}]
\end{aligned}$$

then $v(\Omega) - \sum_{i\in N} \pi^i_{\Omega_i} + \sum_{i\in N} \pi^i_{S_i} = \pi^b + \sum_{i\in F(S)} \pi^i_{S_i} \geq v(S)$.

Step 3: Constraints (5) in **RDLP** are verified.

Given $S \subseteq \Omega_i$ and $i \in N$,

$$\pi^i_{S_i} = p_i^*(S_i) \leq v(\Omega) - v(\Omega \backslash \Omega_i) = p_i^*(\Omega_i) = \pi_i$$

Step 4: Constraint (6) in **RDLP** is verified.

Notice that $\pi^i_{\Omega_i} = p_i^*(\Omega_i) = \pi_i$, thus

$$\begin{aligned}
\pi^b + \sum_{i\in N} \pi^i_{\Omega_i} &= v(\Omega) - \sum_{i\in N} p_i^*(\Omega_i) + \sum_{i\in N} p_i^*(\Omega_i) \\
&= v(\Omega) = V(\Omega)
\end{aligned}$$

Thus, we have proved that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{DLP})$. To show that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{RDLP})$ we need the following last step.

Step 5: $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{RDLP})$

Consider any $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i})) \in sol(\mathbf{RDLP})$. Then, by monotonicity of v and Proposition 2, $(\Omega, \widehat{p}) \in SPE^*$-outcome set, where $\widehat{p}_i(S_i) = \widehat{\pi}^i_{S_i}$ and $\widehat{p}_i(\Omega_i) = \widehat{\pi}_i$. By Lemma 5, $\widehat{p}_i(\Omega_i) \leq v(\Omega) - v(\Omega \backslash \Omega_i) = p_i^*(\Omega_i)$, thus

$$\sum_{i\in N} \widehat{\pi}_i = \sum_{i\in N} \widehat{p}_i(\Omega_i) \leq \sum_{i\in N} p_i^*(\Omega_i) = \sum_{i\in N} \pi_i$$

and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{RDLP})$.

Now, consider S^* such that $v(S^*) = v(\Omega)$. Then by (FS),

$$0 = v(\Omega) - v(S^*) \geq \sum_{i \in N} p_i^*(\Omega_i) - p_i^*(S_i^*) \geq 0$$

thus, $p_i^*(\Omega_i) = p_i^*(S_i^*)$ for all $i \in N$. Moreover, for all $i \in N \backslash F(S^*)$ and all $T_i \subseteq \Omega_i$, $p_i^*(T_i) = 0$, given that

$$\begin{aligned} 0 &\leq v(\Omega \backslash (\Omega_i \backslash T_i)) - v(\Omega \backslash \Omega_i) \leq p_i^*(T_i) \\ &\leq v(\Omega) - v(\Omega \backslash \Omega_i) \leq v(\Omega) - v(S^*) = 0 \end{aligned}$$

Then $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{DLP})$, where $\pi^i_{T_i} = p_i^*(T_i)$, $\pi_i = p_i^*(S_i^*)$ and $\pi^b = v(S^*) - \sum_{i \in F(S^*)} p_i^*(S_i^*)$. That $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{RDLP})$ is proved in the same way as above.

Running the argument of step 2 in reverse yields the equivalence.

■

Proof of Proposition 6:

Step 1: $conv\{x^\sigma(v-c) | \sigma \in \Sigma\} = core(v-c)$.

Let Σ be the set of permutations (orderings) of $N = \{1, 2, ..., n\}$ and let $\sigma \in \Sigma$ be any of its elements. Let P_i^σ be the set of firms which precede firm i with respect to permutation σ, i.e., for all $i \in N$ and $\sigma \in \Sigma$,

$$P_i^\sigma = \{j \in N | \sigma(j) < \sigma(i)\}$$

Define, following Shapley (1971), the marginal contribution vector $x^\sigma(v-c) \in R^n$ of $(v-c)$ with respect to ordering σ by,

$$x_i^\sigma(v-c) = V(P_i^\sigma + i) - V(P_i^\sigma), \text{ for all } i \in N$$

If $(v-c)$ is convex, then the marginal contribution vector $x^\sigma(v-c)$ is positive.

The equality between the sets $conv\{x^\sigma(v-c) | \sigma \in \Sigma\}$ and $core(v-c)$ is given in Driesen 1993.

Step 2: $core(v-c) = P(\Pi)$

Proof of $core(v-c) \subseteq P(\Pi)$. Let $x \in core(v-c)$. Define $\pi^b = 0, \pi_i = x_i$ and $\pi^i_{T_i} = x_i$ for all $i \in N, T_i \subseteq \Omega_i$ It is straightforward that $(\pi^b, (\pi_i), (\pi^i_{T_i}))$ verifies all the constraints of **DLP.** Moreover, by convexity of $(v-c)$,

$$\sum_{i \in N} \pi_i = \sum_{i \in N} x_i = V(N) = (v-c)(\Omega)$$

so that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mathbf{RDLP})$ and component (π_i) cannot be Pareto-dominated by any other solution, hence $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in \Pi$ and $(\pi_i) \in P(\Pi)$.

Proof of $P(\Pi) \subseteq core(v-c)$. Suppose on the contrary that $P(\Pi) \nsubseteq core(v-c)$. Then there exists $z \in P(\Pi)$ such that $z \notin core(v-c)$. By proposition 3, vector z is not Pareto-dominated by any other vector in $P(\Pi)$. However, the equality between the sets $core(v-c)$ and $conv\{x^\sigma(v-c)|\sigma \in \Sigma\}$, jointly with $z \notin core(v-c)$ yields the existence of i_o such that $z_{i_0} > V(N) - V(N \backslash i_0) = (v-c)(\Omega) - (v-c)(\Omega \backslash \Omega_{i_0})$.

On the other hand, by monotonicity of $(v-c)$ and Proposition 3, we have that $(\Omega, p) \in SPE^*$-outcome set, where $p_i(T_i) = z_i + c(T_i)$ for all $i \in N, T_i \subseteq \Omega_i$; and by Lemma 5, $p_i(\Omega_i) \leq v(\Omega) - v(\Omega \backslash \Omega_i)$ for all $i \in N$. Now we obtain

$$\begin{aligned} p_{i_0}(\Omega_{i_0}) &= z_{i_0} + c(\Omega_{i_0}) > (v-c)(\Omega) - (v-c)(\Omega \backslash \Omega_{i_0}) + c(\Omega_{i_0}) \\ &= v(\Omega) - v(\Omega \backslash \Omega_{i_0}) - c(\Omega) + c(\Omega \backslash \Omega_{i_0}) + c(\Omega_{i_0}) \\ &= v(\Omega) - v(\Omega \backslash \Omega_{i_0}) \end{aligned}$$

We conclude that $p_{i_0}(\Omega_{i_0}) > v(\Omega) - v(\Omega \backslash \Omega_{i_0})$ which contradicts Lemma 5. ■

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