ASSESSMENT OF VOTING SITUATIONS: THE PROBABILISTIC FOUNDATIONS*

Annick Laruelle and Federico Valenciano**

WP-AD 2002-22

Corresponding author: Annick Laruelle, Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, Campus de San Vicente, 03080 Alicante, Spain, e-mail: laruelle@merlin.fae.ua.es

Editor: Instituto Valenciano de Investigaciones Económicas, S.A.

Primera Edición Noviembre 2002

Depósito Legal: V-4550-2002

IVIE working papers offer in advance the results of economic research under way in order to encourage a discussion process before sending them to scientific journals for their final publication.

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* The authors want to thank to M. Braham, M. Machover, N. Megiddo and F. Steffen, for their comments, and to J. F. Mertens for pointing out a misstatement, in all cases referred to previous versions of this paper. This research has been supported by the DGES of the Spanish Ministerio de Educación y Cultura, under project PB96-0247, by the Spanish Ministerio de Ciencia y Tecnología, under project BEC2000-0875, and by the Universidad del País Vasco, under project UPV/EHU00031.321-HA-7918/2000. The first author also acknowledges the financial support from a postdoctoral grant from the Basque Government (2000-2001) and the Spanish Ministerio de Ciencia y Tecnología under the Ramón y Cajal Program..

** A. Laruelle: Universidad de Alicante; F. Valenciano: Universidad del País Vasco.

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Abstract

In this paper we revise the probabilistic foundations of the theory of the measurement of 'voting power' either as success or decisiveness. For an assessment of these features two inputs are claimed to be necessary: the voting procedure and the voters' behavior. We propose a simple model in which the voters' behavior is summarized by a probability distribution over all vote configurations. This basic model, at once simpler and more general that other probabilistic models, provides a clear conceptual common basis to reinterpret coherently from a unified point of view different power indices and some related game theoretic notions, as well as a wider perspective for a dispassionate assessment of the power indices themselves, their merits and their limitations.

Keywords: Voting rules, voting power, decisiveness, success, power indices, simple games, probabilistic models.

1 Introduction

The difficult issues raised by the enlargement of the European Union, specially in connection with the EU's institutions' decision-making have reached the public opinion (see, e.g., Galloway (2001)), and power indices have come into focus of renewed scientific interest. There is an open debate about their meaning and their suitability to assess voting situations in general and the EU decision-making in particular. On the one hand, the lack of compellingness from a positive or a normative point of view of the axiomatic foundations of power indices has to be acknowledged. On the other hand, the existence and misuse of several indices without a clear justification is confusing and does not contribute to their credit. Finally, power indices are often criticized (see, e.g., Garrett and Tsebelis (1999, 2001)) on the basis that the only information they take into account is the voting procedure, while the voters' preferences and other contextual relevant information, which clearly influence the role of voters in actual decision making, are ignored. In our opinion most of this sometimes passionate argument is often based on misunderstanding and lack of a clear conceptual basis. This paper intends to provide a clear and simple model that may serve as a conceptual term of reference for a dispassionate assessment of the power indices themselves, their merits and their limitations.

Since the only recently vindicated Penrose (1946) and the later but much more popular Shapley-Shubik (1954) and Banzhaf (1965) indices, there exists a vast literature on power indices and their applications to political science. Apart from the two best known ones, some other power indices[1] and related concepts have been proposed (Rae (1969), Coleman (1971, 1986), Deegan and Packel (1978), Johnston (1978), Holler and Packel (1983), König and Bräuninger (1998)). On the other hand, there are also to be found in the cooperative game theoretic literature some 'solution' concepts, as semivalues (Weber, 1979, 1988), weak (weighted or not) semivalues (Calvo and Santos, 2000) or some coalitional values (Owen (1977, 1982)) that can be seen as extensions of the concept of power index when restricted to simple games (see e.g., Carreras and Magaña (1994), Laruelle and Valenciano (2001b), Carreras, Freixas and Puente (2002)).

There are basically two approaches to deal with power indices and their game theoretic extensions: the axiomatic approach and the probabilistic one. In the first approach, each power index is interpreted as the unique measure embodying a set of properties that characterizes it. This approach has attracted so far much attention in the literature. Since Dubey's (1975) first axiomatization of the Shapley-Shubik index on the domain of simple games and that of Dubey and Shapley (1979) of the Banzhaf index, several axiomatizations have been proposed of these two indices, as well as of some of other power

[1] See Felsenthal and Machover (1998) for a recent critical review.

indices and related game theoretic extensions. However, most of these axiomatizations pay little attention to the compellingness or even to the meaning of the axioms in terms of the voting situations underlying simple games[2].

An alternative approach consists of a direct probabilistic interpretation of the involved concept. This approach received considerable attention in the 70's (see for instance Niemi and Weisberg (1972)), but its appeal seems to have declined in the political science literature (see notwithstanding Straffin (1977, 1982, 1988) and Barry (1980)). While in the game theoretic literature the probabilistic interpretation is disregarded or artificially done in terms of every player's subjective probability distribution over the coalitions she can join.

In this paper we propose a simple model which includes the two separate basic ingredients in a voting situation: the voting rule and the voters. The voting rule specifies when a proposal is to be accepted or rejected depending on the resulting vote configuration. Voters, the second ingredient in a voting situation, are included via their voting behavior, which is summarized by a distribution of probability over the vote configurations. This probability distribution, a black box like ingredient in our model, obviously depends on the preferences of the actual voters over the issues they will have to decide upon, the likelihood of these issues being proposed, the agenda-setting issue, etc. But the minimalistic simplicity of this model, avoiding any further elements in it, has some conceptual advantages. As we will see it allows formulations of a great conceptual transparency and generality, rid of dispensable ingredients or discussable assumptions.

Within this framework, at once simpler and more general than some well-known probabilistic models, as it is shown in the paper, we re-examine the concepts of 'success,' and 'decisiveness' that can be traced a long way back in the literature, as well as some conditional variants. This setting allows a simple and precise reformulation of these concepts as probabilities which depend on the voting rule and the voters' voting behavior. In this way previous purely normative notions are conceptually extended to more general positive/descriptive notions, providing a wider perspective to interpret some power indices (once adequately reformulated and generalized) and related concepts form a common point of view, shedding light on their meaning and relations, and their normative value or their lack of it.

The rest of the paper is organized as follows. Section 2 formalizes the first ingredient in any voting situation: the voting rule. In section 3, we define the primitive ex post versions of the concepts of success and decisiveness. Section 4 incorporates to the model

[2] An exception is Laruelle and Valenciano (2001a), where a transparent characterization of both indices is provided and the lack of compelling arguments to choose any of them on solely axiomatic grounds is stressed.

the voters' behavior. Section 5 provides the ex ante extension of the concepts introduced in section 3, as well as some conditional variations of these concepts. In section 6 the positive/descriptive possibilities of the model are briefly discussed. Section 7 treats the special case in which all vote configurations are considered equiprobable with normative purposes, showing how some power indices emerge as particular cases of the general notions introduced in section 5. In section 8 other power indices, as well as some game-theoretic related concepts are examined in the light of the model. Section 9 addresses the comparison with previous probabilistic models. Section 10 summarizes the main conclusions of the paper.

2 Voting rules

A voting situation is a situation in which a set of voters faces decision-making according to the specifications of a voting procedure. Thus, there are two separate ingredients: the voters and what we will call the voting rule. In this section we concentrate in this second element.

A voting rule is a well-specified procedure to make decisions by the vote of any kind of committee of a certain number of members. If the number of voters is n, the different *seats* will be labelled $1, 2, .., n$, and N will denote this set of labels. Voters will be labelled by their seats' labels. Once a proposal is submitted to the committee, voters will cast votes. A *vote configuration* is a possible or conceivable result of a vote, that lists the vote cast by the voter occupying each seat. We will consider only rules that assimilate any vote different from a 'yes' (abstention included) to a 'no'[3]. Under this assumption there are 2^n possible configurations of votes, and each configuration can be represented by the set of labels of the 'yes'-voters' seats. So, we refer as the *vote configuration* S to the result of a vote where only the voters in S vote 'yes', while those in $N \backslash S$, vote (or are assimilated to) 'no'. The cardinal of S will be denoted by s. Sometimes we will say that a configuration S 'contains i' to mean that i's vote was 'yes', that is, $i \in S$.

An N-*voting rule* is fully specified by the set of vote configurations that would lead to the passage of a proposal. These configurations will be called *winning configurations*. In what follows W will denote the set of winning configurations representing an N-voting rule. It will be assumed that a voting rule satisfies these requirements: (i) $N \in W$; (ii) $\emptyset \notin W$; (iii) If $S \in W$, then $T \in W$ for any T containing S; and (iv) If $S \in W$ then $N \backslash S \notin W$. The last condition prevents the passage of a proposal and its negation if they

[3] See Freixas and Zwicker (2002) for a more general notion of voting rule that admits vote configurations with 'different levels of approval'.

were supported by S and $N \backslash S$, respectively[4].

Let VR_N denote the set of all such N-voting rules, each of them identified with the set W of winning configurations that specifies it. Some particular voting procedures that will be alluded later are the following. W^N will denote the *unanimity rule*, in which the only winning configuration is the unanimous 'yes'. Seat *i's dictatorship* is the voting rule in which the decision always coincides with voter on seat i's (i.e., the *dictator's*) vote: $W^i = \{S \subseteq N : i \in S\}$. We will also refer as a 'null voter's seat' in a voting rule, to a seat such that the result of any vote is never influenced by the vote of the voter sitting on it. Namely, in a procedure W, seat i is a *null voter's seat* if for any S containing i, $S \in W$ if and only if $S \backslash \{i\} \in W$. We will drop i's brackets in $S \backslash \{i\}$ or $S \cup \{i\}$.

3 Success and decisiveness ex post

To speak of success or failure, decisiveness or irrelevance, or any other feature concerning the role played in a voting situation requires voters. Let the voters enter the scene and vote on a given proposal. A vote configuration emerges, and the voting rule prescribes the final outcome, passage or rejection of the proposal. If the proposal is accepted (resp., rejected), only the voters who have voted in favor (resp., against) have had success[5]. Thus, being *successful* means having the outcome -acceptance or rejection- one voted for. We will say that a successful voter has been *decisive* in a vote if her vote was crucial for her success; that is, had she changed her vote the outcome would have been different. This is the basic notion behind a variety of concepts of 'voting power.'

Formally we have the following *ex post boolean* notions. 'Ex post' as dependent on the voting rule used to make decisions and the resulting configuration of votes *after* a vote is cast; and 'boolean' in the sense that there is *no quantification* in these notions, a voter just may or may not be successful or decisive.

Definition 1 *After a decision is made according to an N-voting rule W, if the resulting configuration of votes is S, and $i \in N$,*

(i) Voter i is said to have been successful (for brief, i is successful in (W, S)), if the decision coincides with voter i's vote, that is, iff

$$(i \in S \in W) \text{ or } (i \notin S \notin W). \tag{1}$$

(ii) Voter i is said to have been decisive (for brief, i is decisive in (W, S)), if voter i

[4] In certain cases, e.g., if the rule is used to include issues on the agenda, this condition is not required.

[5] The expression is due to Barry (1980), but the notion can be traced back under different names at least to Rae (1969) (see also Brams and Lake (1978), and Straffin, Davis and Brams (1981)).

was successful and i's vote was critical for it, that is, iff

$$(i \in S \in W \text{ and } S \backslash i \notin W) \text{ or } (i \notin S \notin W \text{ and } S \cup i \in W). \tag{2}$$

The two[6] ex post concepts introduced depend on the resulting vote configuration and the voting rule which prescribes whether such a configuration is winning or not. Can these concepts be defined *ex ante,* that is, before voters cast their vote? If what the voters will vote is known with certainty, the answer is obvious. Otherwise, only in a few cases a partial answer is possible. For instance, a dictator will surely be successful and decisive. And a null voter will never be decisive. But in general, the knowledge of the voting rule is not sufficient to determine ex ante the success and decisiveness of a voter. Indeed a voter's success and decisiveness depend on the voting rule but also on how she and the other voters will vote. In other words, they depend also on all voters' behavior.

4 Voting behavior

In general what voters are going to vote is not known in advance. Nevertheless, an estimation of the likelihood of different vote configurations from the available information is always possible. We assume thus that for any vote configuration S that may arise we know -or at least have an estimate of- the probability $p(S)$ that voters vote in such a way that S emerges. In this way we incorporate into the model the voters' voting behavior via a probability distribution over all possible vote configurations. In other words, the elementary events are the vote configurations in 2^N. As the number of them is finite (2^n), we can represent any such a probability distribution by a map $p : 2^N \rightarrow R$ that associates with each vote configuration S its probability of occurrence $p(S)$. That is, $p(S)$ gives the probability that voters in S vote 'yes', and those in $N \backslash S$ vote 'no'. Of course, $0 \leq p(S) \leq 1$ for any $S \subseteq N$, and $\sum_{S \subseteq N} p(S) = 1$[7].

Let $\mathfrak{P}_N$ denote the set of all such distributions of probability over 2^N. This set can be interpreted as the set of all conceivable probabilistic voting behaviors of N-voters (yes/no voters, in fact, as we assume that there is no abstention) within the present setting. These probabilities permit in principle to reflect the relative proximity of voters' preferences, their relationships, or any contextual information available that conditions their voting behavior, summarizing it in probabilistic terms. It is worth noting that in this model the

[6] Barry (1980) referred also to the successful but irrelevant voters as 'lucky'. That is, a voter i has been lucky in (W, S) iff

$$(i \in S \in W \text{ and } S \backslash i \in W) \text{ or } (i \notin S \notin W \text{and } S \cup i \notin W).$$

The three concepts are obviously related: a successful voter must be either decisive or lucky.

[7] Mind here '$\emptyset$' does not denote the empty event, but the unanimous 'no', so that $p(\emptyset) > 0$ is possible.

event 'voter i votes "yes"' is not necessarily independent of the other voters' votes[8], this is only a particular case within our model.

Now the model is complete: an N-voting situation is specified by a pair (W, p), where $W \in VR_N$ is a voting rule and $p \in \mathfrak{P}_N$ represents a probability distribution over the vote configurations. Mind this second ingredient in the model is a black box probabilistic summary of the voters' behavior. In section 6 we will come back to this point and discuss how to fill in this box either for applied or theoretical purposes.

5 Success and decisiveness ex ante

5.1 The two basic notions

The ex ante version in a voting situation of the concepts introduced in section 4 in their primitive ex post version is now possible. By *ex ante*[9], we mean before the voters cast their votes, but once they occupied their seats. Ex ante, success and decisiveness can be defined as the probability of being successful and decisive, respectively. It suffices to replace in the ex post definitions (1) and (2) the sure configuration S by the random configuration of votes specified by the distribution of probability over the vote configurations p. This yields the following extension of these concepts.

Definition 2 *Let (W, p) be an N-voting situation, where W is the voting rule to be used and $p \in \mathfrak{P}_N$ is the probability distribution over vote configurations, and let $i \in N$:*

(i) Voter i's (ex ante) success is the probability that i is successful:

$$\Omega_i(W,p) := Prob\ \{i\ is\ successful\} = \sum_{S: i \in S \in W} p(S) + \sum_{S: i \notin S \notin W} p(S). \tag{3}$$

(ii) Voter i's (ex ante) decisiveness is the probability that i is decisive:

$$\Phi_i(W,p) := Prob\ \{i\ is\ decisive\} = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} p(S) + \sum_{\substack{S: i \notin S \notin W \\ S \cup i \in W}} p(S) \tag{4}$$

[8] An assumption already considered very irrealistic by Niemi and Weisberg (1972).

[9] In the London Workshop in Voting Power Analysis (11-12/08/02) organized by R. Fara and M. Machover a controversy arised about the use of the term 'a priori' (and consequently 'a posteriori'). In previous versions of the two papers in which this work is based we used this term referring to the situation before the voters cast their votes, but once they occupied their seats, or at least some information about them is in principle available. While the most common use of this term refers to the situation previous to both things and ignoring anything about the voters preferences, relationships, or any possibly relevant contextual information beyond the voting rule itself (see notwithstanding, among others, Owen (1982), Calvo and Lasaga (1997), or Braham and Steffen (2002) for a non that 'radical' use of the term). In order to avoid any confusion and any further controversy we will use the terms 'ex post' and 'ex ante'. Thanks are due to Ian Mc Lean for his suggestion.

These measures[10] provide a precise and rather general formulation of the notions of success and decisiveness based on the primitive ex post notions, with which they are consistent. If restricted versions of all these measures can be traced a long way back in the literature on collective decision-making (as will be seen in sections 7 and 8), so far the probability distribution over the vote configurations has not been considered as an (in general) independent input with the generality (and absence of further ingredients) with which it is considered here. Usually such a distribution of probability is hidden or only implicit in the definition of some notions related with 'power', or burdened with additional ingredients and assumptions.

Note that strictly speaking i's decisiveness depends only on the behavior of the other voters, not on hers. To see this voter i's decisiveness can be rewritten as

$$\Phi_i(W,p) = \sum_{\substack{S:i\in S\in W \\ S\setminus i\notin W}} (p(S) + p(S\setminus i)). \tag{5}$$

Observe that for each S, $p(S) + p(S\setminus i)$ is the probability of all voters in $S\setminus i$ voting 'yes' and those in $N\setminus S$ voting 'no'. In this case, whatever voter i's vote, she would be decisive. While Ω_i depends on the behavior of all the voters. Thus there is no general way to derive one concept from the other, the only relation in general being the obvious $\Phi_i(W,p) \leq \Omega_i(W,p)$, as well as Barry's equation: 'Success' = 'Decisiveness' + 'Luck,' which remains valid in a much more precise and general version. Namely, for any voting rule W, any probability distribution p, and any voter i, we have

$$\Omega_i(W,p) = \Phi_i(W,p) + \Lambda_i(W,p).$$

5.2 Conditional variants

The precise probabilistic setting in which these notions stand permits to address the accurate formulation of further specific questions for a given voting situation (W,p). For instance, if voter i is sure to vote in favor of (or against) the proposal, the *conditional* probabilities of success and decisiveness can be evaluated. Alternatively, success and decisiveness can be defined conditionally to the acceptance or to the rejection of the proposal. The corresponding conditional probability gives the answer to each of the following questions:

[10] Similarly, voter i's (ex ante) 'luck' would be the probability that i is lucky, that is:

$$\Lambda_i(W,p) := \sum_{\substack{S:i\in S \\ S\setminus i\in W}} p(S) + \sum_{\substack{S:i\notin S \\ S\cup i\notin W}} p(S).$$

Q.1: Which is voter i's conditional probability of success (resp., decisiveness), given that voter i votes in favor (resp., against) of the proposal?

Q.2: Which is voter i's conditional probability of success (resp., decisiveness), given that the proposal is accepted (resp., rejected)?

The conditional probabilities which answer any of these questions are given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \tag{6}$$

where A may stand for 'voter i is successful/decisive' and B stands either for 'voter i votes "yes"/"no"', or 'the proposal was accepted/rejected'[11]. This makes eight possible conditional probabilities which answer the previous questions. Of course, the framework allows for other questions involving different conditions (e.g., conditional to 'i and j voted the same'). We restrict to these ones because, as we will see in section 7, some power measures proposed in the literature can be reinterpreted as one of these conditional probabilities for a particular probability distribution. A bit of notation is necessary. We will superindex the measures -Ω_i or Φ_i - when they represent conditional probabilities. The superindex '$i+$' (resp., '$i-$') will refer to the condition 'given that i votes "yes"(resp., "no")'. So the answers to Q.1 are given by Ω_i^{i+}, Φ_i^{i+}, Ω_i^{i-} and Φ_i^{i-}, respectively. The superindex 'Acc' (resp., 'Rej') will refer to the condition 'given that the proposal is accepted (resp., rejected)'. Thus the answers to Q.2 are given by Ω_i^{Acc}, Φ_i^{Acc}, Ω_i^{Rej} and Φ_i^{Rej}, respectively. As an illustration, we formulate explicitly two of them. Denoting

$$\gamma_i(p) := Prob\ \{i \text{ votes 'yes'}\} = \sum_{S: i \in S} p(S),$$

$$\alpha(W,p) := Prob\ \{\text{acceptance}\} = \sum_{S: S \in W} p(S). \tag{7}$$

Voter i's conditional probability of being decisive given that voter i votes in favor of the proposal, is given by:

$$\Phi_i^{i+}(W,p) := Prob\ \{i \text{ is decisive} \mid i \text{ votes 'yes'}\} = \frac{\sum\limits_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} p(S)}{\gamma_i(p)}. \tag{8}$$

Voter i's conditional probability of success given that the proposal is accepted, is given by:

$$\Omega_i^{Acc}(W,p) := Prob\ \{i \text{ is successful} \mid \text{the proposal is accepted}\} = \frac{\sum\limits_{S: i \in S \in W} p(S)}{\alpha(W,p)}.$$

[11] Of course, conditional probabilities only make sense if $p(B) \neq 0$. This will be implicitly assumed whenever we refer to any of these conditional measures.

The following table summarizes the ten (unconditional and conditional) variants:

Condition:	*none*	*i votes 'yes'*	*i votes 'no'*	*acceptance*	*rejection*
Success	Ω_i	Ω_i^{i+}	Ω_i^{i-}	Ω_i^{Acc}	Ω_i^{Rej}
Decisiveness	Φ_i	Φ_i^{i+}	Φ_i^{i-}	Φ_i^{Acc}	Φ_i^{Rej}

Table 1

They are related by

$$\Omega_i(W,p) = \gamma_i(p)\ \Omega_i^{i+}(W,p) + (1-\gamma_i(p))\ \Omega_i^{i-}(W,p),$$

$$\Phi_i(W,p) = \gamma_i(p)\ \Phi_i^{i+}(W,p) + (1-\gamma_i(p))\ \Phi_i^{i-}(W,p), \tag{9}$$

$$\Omega_i(W,p) = \alpha(W,p)\ \Omega_i^{Acc}(W,p) + (1-\alpha(W,p))\ \Omega_i^{Rej}(W,p),$$

$$\Phi_i(W,p) = \alpha(W,p)\ \Phi_i^{Acc}(W,p) + (1-\alpha(W,p))\ \Phi_i^{Rej}(W,p).$$

Mind that *Prob* $\{i$ votes 'no'$\} = 1-\gamma_i(p)$, and *Prob* $\{$rejection$\} = 1-\alpha(W,p)$.

In section 7 we will see how seven out of these ten variants (eight out of eleven if we include $\alpha(W,p)$[12]) are related with some power indices of which they can be interpreted as the natural conceptual extension for arbitrary behaviors. In particular the three measures Φ_i, Φ_i^{i+}, and Φ_i^{i-}, or better their particularization for some particular explicit or implicit probability distribution are some times confused as equivalent, which in general it is not true. The following proposition characterizes the behaviors for which the three measures coincide.

Proposition 1 *For a distribution of probability $p \in \mathfrak{P}_n$ the three measures $\Phi_i(W,p)$, $\Phi_i^{i+}(W,p)$ and $\Phi_i^{i-}(W,p)$ coincide for every i and every voting rule W, if and only if the vote of every voter is independent from the vote of the remainder voters.*

Proof. First note that the two conditional measures make sense only if the case where any voter i votes 'yes' (or 'not') with probability zero are excluded. Thus, we assume $0 < \gamma_i(p) < 1$, for all i. Now by (9), if any two measures coincide, the third one will

[12]In Laruelle and Valenciano (2002) it is shown a general result related to TU games that restricted to simple games yields an interesting interpretation of $\alpha(W,p)$ as the 'generalized' potential (or $\frac{\alpha(W,p)}{\gamma_i(p)}$ as the 'traditional' Hart and Mas-Colell's (1989) potential).

coincide too. Thus it is enough to prove that $\Phi_i^{i+}(-,p) = \Phi_i^{i-}(-,p)$ for all i if and only if p satisfies the independence condition above. We write γ_i instead of $\gamma_i(p)$ for brief.

Sufficiency: Assume that every voter independently votes 'yes' with a certain probability. Then $p(S) = \prod\limits_{i \in S} \gamma_i \prod\limits_{j \in N \backslash S} (1-\gamma_j)$, for any $S \subseteq N$, from which it follows immediately that $\frac{p(S)}{\sum\limits_{T:i\in T} p(T)} = \frac{p(S\backslash i)}{1-\sum\limits_{T:i\in T} p(T)}$ for all $S \neq \emptyset$, and all $i \in S$. Consequently, from formulae (8) and the similar one giving $\Phi_i^{i-}(W,p)$ it follows immediately that $\Phi_i^{i+}(W,p) = \Phi_i^{i-}(W,p)$ for any procedure W.

Necessity: Now assume $\Phi_i^{i+}(W,p) = \Phi_i{}^{i-}(W,p)$ for all i and any procedure W. Let us see that this implies $p(S) = \prod\limits_{i\in S} \frac{\gamma_i}{1-\gamma_i} p(\emptyset)$ for all $S \neq \emptyset$. For it, take the unanimity rule $W^N = \{N\}$. The coincidence of both measures for this procedure implies $p(N) = \frac{\gamma_i}{1-\gamma_i} p(N\backslash i)$ for all i. Now, for any $S \subseteq N$ s.t. $s \geq 2$ and any $i \in S$, take $W = \{T \subseteq N : S\backslash i \subseteq T \text{ and } t \geq s\}$. The coincidence of both measures for this procedure entails that $p(S) = \frac{\gamma_i}{1-\gamma_i} p(S\backslash i)$. Finally, taking i's dictatorship $W^i = \{T \subseteq N : i \in T\}$, the coincidence of both measures together with the previous equalities yield that $p(\{i\}) = \frac{\gamma_i}{1-\gamma_i} p(\emptyset)$. Thus, we have that $p(S) = \frac{\gamma_i}{1-\gamma_i} p(S\backslash i)$ for all $S \neq \emptyset$, and all $i \in S$. Then for any $S \neq \emptyset$ we can write $p(S) = \prod\limits_{i\in S} \frac{\gamma_i}{1-\gamma_i} p(\emptyset)$. Substituting these equations in $\sum\limits_{S\subseteq N} p(S) = 1$, we get

$$\sum_{S\subseteq N} p(S) = p(\emptyset) + \sum_{S\neq\emptyset} p(S) = p(\emptyset) + \sum_{S\neq\emptyset} \prod_{i\in S} \frac{\gamma_i}{1-\gamma_i} p(\emptyset) = 1,$$

that is,

$$p(\emptyset) = \frac{1}{1 + \sum_{S\neq\emptyset} \prod\limits_{i\in S} \frac{\gamma_i}{1-\gamma_i}} = \frac{\prod\limits_{i\in N} (1-\gamma_i)}{\sum_{S\subseteq N} \prod\limits_{i\in S} \gamma_i \prod\limits_{j\in N\backslash S} (1-\gamma_j)} = \prod_{i\in N} (1-\gamma_i).$$

Where the last equality results from the denominator being 1 (this, as $0 < \gamma_i < 1$ for all i, is obvious for $n = 1$ or 2, and easy to prove by induction for all n). Thus we have that $p(S) = \prod\limits_{i\in S} \frac{\gamma_i}{1-\gamma_i} p(\emptyset) = \prod\limits_{i\in S} \gamma_i \prod\limits_{j\in N\backslash S} (1-\gamma_j)$ for any $S \subseteq N$. Thus the vote of every voter is independent from the vote of the rest. ■

In other words, this coincidence holds only for the particular class of probabilistic voting behaviors in which every voter independently votes 'yes' with a certain probability. This includes, as we will see, the Banzhaf index but not the Shapley-Shubik index.

Another question is whether different behaviors can lead to the same measure of decisiveness. The following proposition gives the necessary and sufficient conditions for this to be so for each of the three measures of decisiveness.

Proposition 2 *Let p and $p' \in \mathfrak{P}_N$, then* *(i)* $\Phi_i(W,p) = \Phi_i(W,p')$ *for all* i *and any voting rule* W *if and only if*

$$p'(S) = p(S) + (-1)^{s+1}(p(\emptyset) - p'(\emptyset)) \text{ for all } S \neq \emptyset.$$

(ii) $\Phi_i^{i+}(W,p) = \Phi_i^{i+}(W,p')$ *for all* i *and any voting rule* W *if and only if*

$$\frac{p(S)}{1-p(\emptyset)} = \frac{p'(S)}{1-p'(\emptyset)} \text{ for all } S \neq \emptyset.$$

(iii) $\Phi_i^{i-}(W,p) = \Phi_i^{i-}(W,p')$ *for all* i *and any voting rule* W *if and only if*

$$\frac{p(S\backslash i)}{1-p(N)} = \frac{p'(S\backslash i)}{1-p'(N)} \text{ for all } S \neq \emptyset.$$

Note that in (ii) (resp., in (iii)) for Φ_i^{i+} (resp., Φ_i^{i-}) to make sense it must be assumed that for all i, $\gamma_i(p) > 0$ (resp., $\gamma_i(p) < 1$), which entails $p(\emptyset) < 1$ (resp., $p(N) < 1$). We omit the details of the proof, whose basic idea is as follows. For (i) it is easy to see that behavior influences decisiveness via the sum of the probabilities of each configuration and the one resulting from it when a voter changes her vote from 'yes' to 'no'. Therefore different distributions satisfying this condition lead to the same measure. For (ii) the point is that what matters for the conditional measures Φ_i^{i+} are the probabilities of vote configurations where at least one voter votes 'yes'. The probability of a unanimous 'no' does not affect these measures, therefore the probability of this configuration can be modified and rescale proportionally the probability of the others without modifying them. Finally, for (iii), the situation is similar just replacing the configuration $\emptyset$ by the configuration N. Observe that, for any of the three measures, no two different probability distributions for which the unanimous 'no' (unanimous 'yes' for Φ_i^{i-}) has zero probability have the same associated measure.

6 Positive versus normative approach

The basic concepts given by (3) and (4) in Definition 2, as well as all the conditional variations of them considered, can in principle be used for a *positive* or *descriptive* evaluation of a voting situation. For such an evaluation the voting rule is not sufficient, an estimate of the voters' voting behavior is needed too. In our basic formulations this second ingredient is summarized by a probability distribution over vote configurations. This 'black box' can be filled from available data for empirical or applied purposes, or by enriching the model for theoretical purposes. In the first case, ex ante there is not such a general thing as 'the best positive or descriptive measure' of actual or de facto power in any of the senses specified so far, beyond the general formulae based on the two inputs. In every particular real world voting situation all that can be said is that the better the estimate of the probability distribution over vote configurations that best suits the case, the better the measure of actual decisiveness. This entails the search of data for an estimate of this probability distribution over voting configurations that better summarizes the voters' behavior[13]. An interesting approach could be using empiric probabilities based on the frequencies of voting configurations. At the theoretical level, definitions (3) and (4), and the conditional variations considered, provide a basic conceptual set up open to the connection with more complex models involving voters' preferences or other contextual information, shared or not by all voters, or models in which voters have 'spatial preferences', in which this probability can be endogenously generated (see e.g., Napel and Widgrén (2002) for a more sophisticated model consistent with this one).

It is worth remarking that the general measures considered so far are conceptually beyond Garrett and Tsebelis' criticism of power indices under the basis that the voters' preferences, and any other relevant contextual information are ignored. To illustrate this point let us reconsider the example that Garrett and Tsebelis (1999) used to illustrate their claim. They consider a 7-voters voting rule where a proposal is passed if it has the support of at least 5. They assume that voters are located on a real line so that only connected and minimal winning configurations occur, and all of them are equiprobable. Under these assumptions, they claim that a 'more realistic power index' should give $\frac{1}{15}$ for voters 1 and 7, $\frac{2}{15}$ for voters 2 and 6, and $\frac{1}{5}$ for voters 3, 4 and 5, respectively. In fact, the conditions specified in the model yield the following probability distribution

$$p^{GT}(S) = \begin{cases} \frac{1}{3} & \text{if } S \in \{\{1,2,3,4,5\},\{2,3,4,5,6\},\{3,4,5,6,7\}\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the probability of being decisive for this voting situation (W^{GT}, p^{GT}), where the voting rule is $W^{GT} = \{S : s \geq 5\}$, is given by $\Phi(W^{GT}, p^{GT})$, that is:

$$\Phi_1 = \Phi_7 = \frac{1}{3}, \quad \Phi_2 = \Phi_6 = \frac{2}{3}, \quad \Phi_3 = \Phi_4 = \Phi_5 = 1.$$

Denoting $\widetilde{x}$ the normalization of any vector $x \in R^n$

$$\widetilde{x} := \frac{x}{\sum_{i \in N} x_i},$$

we get that $\tilde{\Phi}$ is Garrett and Tsebelis' proposed normalized connected power index. Thus Garrett and Tsebelis's little story can be accommodated easily in our conceptual

[13] An earlier version of this paper raised a sceptical comment about the difficulties of assessing the probabilities of 2^n different possible events. Nevertheless in real life, where often only a few configurations are likely, such assessements are more or less roughly done all the time. In a formal (though completely different) framework, Calvo and Lasaga (1997) obtained from political analysts an assessment of the probability of every two parties in the Spanish Parliament to agree.

framework[14]. Of course, we do not claim that the simple model presented in this paper accounts for everything that can be of interest about any real world voting situation. This point is discussed in the last section.

In opposition to the positive/descriptive point of view considered so far, there is the *normative* point of view. This is the case when one is concerned with the normative issues that arise in the assessment of a voting situation or the design of a voting rule, irrespective of which voters occupy the seats. For this purpose, the particular personality or preferences of the voters, that evidently influences their behavior, *should not* be taken into account. In this case we are at a logical deadlock: no measurement seems possible without a probability distribution over vote configurations, but a crucial part of the information relevant to estimate this probability has to be ignored. What can be done? Here only the analyst's or the designer's choice, consistent with the situation and the aim, can solve the deadlock. This is the point where the meaning of the term 'a priori', understood as the right amount of information to be taken into account for normative purposes, is critical. Different authors in different cases have used the term with different meanings. For instance Owen (1977, 1982) in the very title of his papers refers to 'a priori unions', meaning the blocks formed by voters before casting any vote (see section 8). Calvo and Lasaga (1997) refer to 'a priori ideological compatibility' of any two parties. In more general terms Braham and Steffen (2002) argue in support of a notion of 'a prioricity' that ignores the voters' preferences but incorporates the 'structure' that conditions their behavior. In the next section we examine a particular choice that stands out on its own specificity.

7 Assessment of the voting rule itself

A way out of the difficulty discussed in the last paragraph of the previous section consists of assuming equally probable all configurations of votes:

$$p^*(S) := \frac{1}{2^n} \quad \text{for all configuration } S \subseteq N.$$

As is well-known this is equivalent to assuming that each voter, independently from the others, votes 'yes' with probability 1/2, and votes 'no' with probability 1/2. This choice is consistent with the most basic normative aim according to which any information beyond the rule itself should be ignored. Note that also from a positive point of view, p^* is the natural starting point in case of actual absolute ignorance about the voters and the

[14]Mind that while the vector $\Phi(\mathcal{W}^{GT}, p^{GT})$ gives every voter's probability of being decisive in the decision-making by voting rule $\mathcal{W}^{GT}$ when voters' behavior is represented by p^{GT}, its normalization destroys its interpretation, so that $\tilde{\Phi}$ has no clear meaning.

context[15]. Although we do not share the dogmatic view according to which the only legitimate use of the term 'a priori' is this radical one, it is clear that this extreme case deserves attention on its own right. It makes sense when the objective is *not to assess a voting situation, but the voting rule itself.*

In fact, as we will presently see, some 'power indices' can be seen as the particularization of some of the measures introduced in section 5 for this specific probability distribution. This is the case of Rae's (1969) 'expected correspondence between individual values and collective choices', the (non normalized) Banzhaf (1965) index and the Coleman (1971, 1986,) indices, and even the more recent König and Bräuninger's (1998) 'inclusiveness' index. Thus our model provides a common conceptual basis for the interpretation and the normative justification of these power indices. But, and this is also significant, *not all* power indices in the literature fit in this common setting, as we will see in section 8.

Rae index

Rae (1969) studies the anonymous[16] voting rule that maximizes the correspondence between a single anonymous individual vote and those expressed by collective policy, assuming that each voter, independently from the others, votes 'yes' with probability 1/2, and votes 'no' with probability 1/2[17]. Dubey and Shapley (1979) suggest that the index can be generalized to any voting rule and for any voter. This leads to what can be referred to as the Rae index, given by

$$Rae_i(W) := \sum_{S:i\in S\in W} \frac{1}{2^n} + \sum_{S:i\notin S\notin W} \frac{1}{2^n}.$$

That is, Rae index is but the success (3) for the particular distribution p^*:

$$Rae_i(W) := \Omega_i(W, p^*).$$

Banzhaf index

Banzhaf's (1965) original or 'raw' index to assess the relative (i.e., ratio of) 'power' (as decisiveness) for a seat i and voting rule W is given by:

[15] Felsenthal and Machover (1998, p. 38) refer to the so-called 'Principle of Insufficient Reason' to justify this distribution of probability.

[16] That is, one in which the winning or losing character of a configuration only depends on its size.

[17] In fact he makes three assumptions: (i) The probability that one member will support (or oppose) a proposal is independent of that probability for any other member. (ii) The probability that each member will support any proposal is exactly one-half, and the probability that he will oppose it is also one-half. (iii) The probability that no member supports the proposal is zero. But (iii) must be dropped for under assumptions (i) and (ii), the probability that no one supports the proposal is necessarily $1/2^n$.

$$rawBz_i(W) := \text{number of winning configurations in which } i \text{ is decisive},$$

and Owen (1975) (see also Dubey and Shapley, 1979) proposed the following relativization of this index as a ratio

$$Bz_i(W) = \frac{\text{number of winning configurations in which } i \text{ is decisive}}{\text{total number of voting configurations containing } i}.$$

As a voting configuration containing i means one in which i votes 'yes', it can be easily seen that $Bz_i(W) = \Phi_i^{i+}(W, p^*)$. Moreover, in view of Proposition 1, we have

$$Bz_i(W) = \Phi_i(W, p^*) = \Phi_i^{i+}(W, p^*) = \Phi_i^{i-}(W, p^*).$$

This provides three different interpretations of the Banzhaf index as an expectation of being decisive, and inverting the point of view, Φ_i, Φ_i^{i+}, and Φ_i^{i-} are three different extensions of the Banzhaf index for arbitrary voting behaviors.

Coleman indices

Coleman (1971, 1986) defines, also in terms of ratios, three different indices. The 'power of a collectivity to act', that measures the easiness to make decisions by means of a voting rule W, given by the ratio

$$A(W) = \frac{\text{number of winning configurations}}{\text{total number of voting configurations}}.$$

Voter i's Coleman index 'to prevent action' (Col_i^P) is given by the ratio

$$Col_i^P(W) = \frac{\text{number of winning configurations in which } i \text{ is decisive}}{\text{total number of winning configurations}}.$$

While voter i's Coleman index 'to initiate' action (Col_i^I) is given by the ratio

$$Col_i^I(W) = \frac{\text{number of losing configurations in which } i \text{ is decisive}}{\text{total number of losing configurations}}.$$

Observe that the only input necessary to determine any of these three indices is the voting rule: no distribution of probability enters explicitly their definitions. But reinterpreting them in probabilistic terms, the implicit assumption behind these indices is that all vote configurations are equally probable. Then we have the following conclusions about the meaning of the Coleman indices:

$$A(W) = \alpha(W, p^*) = Prob\ \{\text{acceptance}\} = \sum_{S:S \in W} p^*(S),$$

$$Col_i^P(W) = \Phi_i^{Acc}(W, p^*), \tag{10}$$

$$Col_i^I(W) = \Phi_i^{Rej}(W, p^*). \tag{11}$$

König-Bräuninger's inclusiveness index

Finally, recently König and Bräuninger (1998) define voter i's 'inclusiveness' as the ratio of winning configurations containing i, that for a voting rule W is given by:

$$KB_i(W) = \frac{\text{number of winning configurations containing } i}{\text{total number of winning configurations}}.$$

Again a notion that can be generalized to arbitrary voting behaviors as $\Omega_i^{Acc}(W, p)$, and that in our setting can be redefined as

$$KB_i(W) := \Omega_i^{Acc}(W, p^*).$$

Summary

The following table summarizes the relations of these 'power indices', some of them already classical, and the general model presented in this paper. Table 1, *for the probability distribution p^* that assigns the same probability to all voting configurations*, becomes

Condition:	*none*	*i votes 'yes'*	*i votes 'no'*	*acceptance*	*rejection*
Success	$\Omega_i = Rae_i$	Ω_i^{i+}	Ω_i^{i-}	$\Omega_i^{Acc} = KB_i$	Ω_i^{Rej}
Decisiveness	$\Phi_i = Bz_i$	$\Phi_i^{i+} = Bz_i$	$\Phi_i^{i-} = Bz_i$	$\Phi_i^{Acc} = Col_i^P$	$\Phi_i^{Rej} = Col_i^I$

Table 2

Inverting the point of view, the functions dependent on the voting situation (W, p): Φ_i, Φ_i^{i+}, Φ_i^{i-}, α, Φ_i^{Acc}, Φ_i^{Rej}, Ω_i and Ω_i^{Acc}, *for arbitrary probability distributions, can be seen as the natural positive/descriptive generalizations of the purely normative Banzhaf's, Coleman's, Rae's and König-Bräuninger's indices.*

Equalities (10) and (11) show clearly the difference between the Coleman indices and the Banzhaf index, often mistakenly confused. Both measure decisiveness assuming all vote configurations equally probable. But Banzhaf index measures decisiveness non conditionally (or conditionally to i's positive or negative vote indistinctly), while Coleman indices measure decisiveness conditionally to the acceptance ($Col_i^P(W)$) or the rejection ($Col_i^I(W)$) of the proposal. The origin of the confusion between these indices is due to the fact that their normalizations coincide, giving rise to the so-called 'Banzhaf-Coleman' index. In formula, denoting $\widetilde{x}$ the normalization of any vector $x \in R^n$, we have the following relation for any voting rule W,

$$\widetilde{Bz}_i(W) = \widetilde{Col}_i^P(W) = \widetilde{Col}_i^I(W).$$

This coincidence only advocates against the common practice of normalizing these indices, for, along with the loss of information this normalization entails, it makes them lose their probabilistic interpretation. Mind that in general for arbitrary probability distributions the normalizations of $\Phi_i(W,p)$, $\Phi_i^{Acc}(W,p)$, and $\Phi_i^{Rej}(W,p)$ do not coincide.

Note also that the relation that Dubey and Shapley (1979) establish between the Rae index and the Banzhaf index relies on the assumption that all vote configurations are equiprobable[18]. As mentioned in section 5, in general success and decisiveness are not directly related.

8 Other power indices and game theoretic related notions

In this section we examine whether other power indices, as well as some related game theoretic notions, fit or not into the model. That is, whether they can be interpreted as the probability of being decisive for any probability distribution.

Shapley-Shubik index

For a given decision procedure W, the Shapley-Shubik (1954) index, for each voter i, is given by

$$Sh_i(W) = \sum_{\substack{S:i\in S\in W\\ S\setminus i\notin W}} \frac{(n-s)!(s-1)!}{n!}.$$

As the Banzhaf index, it can be seen as a probabilistic measure of decisiveness, either unconditional ($\Phi_i(\text{-},p)$) or conditional ($\Phi_i^{i+}(\text{-},p)$ or $\Phi_i^{i-}(\text{-},p)$), but unlike the Banzhaf index, this is only so *for different probability distributions* over vote configurations in every case.

Proposition 3 *(i) $\Phi_i(W,p) = Sh_i(W)$ for all i and any voting rule W, if all the vote configurations' sizes (from 0 to n) are equally probable, and all configurations of the same size are equally probable. That is, if*

$$p(S) = \frac{1}{n+1}\,\frac{1}{\binom{n}{s}} \quad \textit{for all } S\subseteq N.$$

(ii) $\Phi_i^{i+}(W,p) = Sh_i(W)$ for all i and any $W\in VR_N$, if a configuration ($\neq\emptyset$) is chosen like this: a size s from 1 to n is chosen with probability inversely proportional to s, then a configuration of size s is chosen at random. That is, if $p(\emptyset)=0$, and

$$p(S) = \frac{\frac{1}{s}}{\sum\limits_{t=1}^{n}\frac{1}{t}}\,\frac{1}{\binom{n}{s}} \quad \textit{for any } S\neq\emptyset.$$

[18] See also Straffin, Davis and Brams (1981).

(iii) $\Phi_i^{i-}(W,p) = Sh_i(W)$ *for all* i *and any* $W \in VR_N$*, if a configuration* ($\neq N$) *is chosen like this: a size* s *from* 0 *to* $n-1$ *is chosen with probability inversely proportional to* $n-s$*, then a configuration of size* s *is chosen at random. That is, if* $p(N)=0$*, and*

$$p(S) = \frac{\frac{1}{n-s}}{\sum_{t=1}^{n} \frac{1}{t}} \frac{1}{\binom{n}{s}} \quad \textit{for any } S \neq N.$$

Mind that all the 'if"s in Proposition 3 would become 'if and only if', if in the three cases the specific probability distribution is replaced by the family of probability distributions that yield the same Φ_i, Φ_i^{i+}, and Φ_i^{i-} (for all i) respectively, that can easily be generated by means of Proposition 2. Note that the stories for the probability distributions in (ii) and (iii) look rather unfamiliar. But, although the Shapley-Shubik index (as some cooperative game theoretic 'solutions', as we presently will see) fit as a particular case of any of the three variations of decisiveness according to the approach considered here, we cannot find any convincing arguments from a normative point of view in favor of any of these very particular probability distributions, and consequently in favor of the Shapley-Shubik index as a normative measure of decisiveness in the sense considered here. As to its suitability in voting situations in which, underneath the voting surface, some 'spoils' were at stake, so that the so-called 'P-power' (Felsenthal and Machover, 1998) were the relevant issue, we refer the reader to the concluding remarks.

Other power indices

Deegan and Packel (1978) and Holler and Packel (1983) introduced two new indices that rely on the concept of minimal winning configuration (m. w. c.) of votes. A winning configuration S is *minimal* if it does not contain properly any other winning configuration. Let $M(W)$ and $M_i(W)$ denote the sets of all m. w. c., and the set of m. w. c. containing i, respectively, and let $m(v)$ and $m_i(W)$, respectively, denote their number. For a voting rule W, voter i's Deegan-Packel index is given by

$$DP_i(W) = \frac{1}{m(W)} \sum_{S \in M_i(W)} \frac{1}{s}.$$

Deegan and Packel (1978, p. 114) justify their index on three assumptions regarding the behavior of the voters that in terms of the model considered here can be reworded as follows: Only minimal winning configurations will emerge, all of them are equally probable, and the members of the resulting configuration will divide the 'spoils' equally.'

Holler and Packel (1983) argue that the third assumption implies a 'private good approach', and opposes a 'public good' approach, substituting the third assumption by this: All voters in a minimal winning coalition get the undivided coalition value. Then we get what Laruelle (1998) refers to as the *non-normalized* Holler-Packel index, that in the notation used here is given by

$$HP_i(W) = \frac{1}{m(W)} \sum_{S \in M_i(W)} 1 = \frac{m_i(W)}{m(W)}.$$

The Holler-Packel index is the normalization this vector, that is,

$$\widetilde{HP}_i(W) = \frac{HP_i(W)}{\sum_{k \in N} HP_k(W)} = \frac{m_i(W)}{\sum_{k \in N} m_k(W)}.$$

In fact, the first two assumptions (common to both indices) specify the following probability distribution (but mind it depends exclusively on the voting rule W!):

$$p_W(S) = \begin{cases} \frac{1}{m(W)} & \text{if } S \text{ is a minimal winning configuration} \\ 0 & \text{otherwise.} \end{cases}$$

Thus for this probability distribution results:

Proposition 4 *For every voting rule W, the non normalized Holler-Packel is given by $HP_i(W) = \Phi_i(W, p_W)$ (for the above described p_W)*[19].

But observe that, properly speaking, not even the non-normalized Holler-Packel index fits into the general definition (4) of decisiveness because the probability distribution is determined by the rule itself. Note also that the normalization that yields the Holler-Packel index destroys this probabilistic interpretation. With respect to the Deegan-Packel index, it should be stressed that the 'distribution of cake' ingredient of the third assumption is completely inconsistent with the approach considered here. The same can be said about Holler and Packel's reinterpretation of the cake as a public good.

Laver (1978) criticizes the power indices and claims that: 'it is clear that a party's power will be greater if it is the only destroyer of a particular coalition than if that honor is shared with a number of others.' In response to Laver's argument, Johnston (1978) proposes a modification of the normalized Banzhaf index. Namely, if $\varkappa(S)$ denotes the number of decisive voters in a winning configuration S, Johnston index is the result of normalizing the vector

$$\sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W \\ \varkappa(S) \neq 0}} \frac{1}{\varkappa(S)}.$$

[19] As M. Machover pointed out, alternatively the non normalized Holler-Packel can be accomodated in this model as the conditional measure of decisiveness under p^*, for the condition 'given that a m.w.c. will form', that is, $HP_i(W) = \Phi_i^{m.w.c.}(W, p^*)$.

We see no way to provide any meaning to this index from the point of view provided by our model.

Coalitional values and other game-theoretic extensions

A coalitional value is an extension of the concept of value for TU games in which, apart from the game itself, a coalition structure is taken also as an input. A *coalitional structure* in a TU game is a partition of the set of players into disjoint coalitions that is interpreted as a form of ex ante union into subgroups of players. Owen (1977, 1982) proposed extensions of the Shapley value (1977) and of the Banzhaf index (1982) (of the 'Banzhaf-Coleman index' in his terms) to these situations. In the context of voting, ex ante unions arise naturally (parties, blocks, etc.). Again our general formulation provides a framework to deal with these situations in which a coalition structure constraints the vote configurations. The natural treatment consists of restricting the class of probability distributions to those that assign probability zero to those configurations that 'break' any coalition in this structure. Note that from a positive/descriptive point of view any further narrowing of the class of probability distributions could only be justified if based on actual data about the situation under consideration. From this point of view, the mechanical restriction to simple games of any of the coalitional values in the literature of TU games, based on purely axiomatic grounds, as a probabilistic measure of decisiveness lacks justification. In view of the lack of compelling arguments in support of the Shapley-Shubik index form the point of view of this approach, we will not deal with the coalitional extensions of this index. Nevertheless, we have the following elegant statement relating Owen's (1982) coalitional value and ex ante decisiveness.

Proposition 5 *Let $\Psi_i(W, \mathcal{B})$ denote the Owen's (1982) coalitional index of a voter $i \in B_j \in \mathcal{B}$, for a voting rule W and a coalitional structure $\mathcal{B}$, then*

$$\Psi_i(W, \mathcal{B}) = \Phi_i(W, p_j^{\mathcal{B}}).$$

where $p_j^{\mathcal{B}}$ denotes the distribution that assigns the same probability to all configurations that do not break any $B_k \neq B_j$, and zero to those which break any $B_k \neq B_j$.

Note that $\Psi_i(W, \mathcal{B})$ only partly fits general formulation (4) because the probability distribution in $\Phi_i(W, p_j^{\mathcal{B}})$ depends on which block voter i belongs to. Namely, for any $i \in B_j$ it is assumed that all blocks *but* B_j act as blocks (i.e., the vote does not split within any of these blocks) and every of these blocks votes 'yes' with probability 1/2 and 'no' with probability 1/2. While *within* B_j all vote configurations are equally probable. Thus, we have again a conditional variation of $\Phi_i(W, p^*)$, but a more complex one, as dependent on $\mathcal{B}$ and on which block the voter belongs to. This provides an interesting example of an 'a priori' (according Owen's own terms) assessment in which some information (not the same for voters in different blocks!) beyond the rule itself is taken into account: the coalitional structure. But in a very particular way: it is taken as part of the environment of a voter in a block (all the others will act as blocks) to assess the a priori decisiveness of every voter within her block given that context.

Finally, there are a variety of 'solution' concepts in cooperative game theory, as semivalues (Weber, 1979, 1988) and weak (weighted or not) semivalues (Calvo and Santos, 2000) that can be seen as generalizations of the concept of power index when restricted to simple games. All these notions were introduced axiomatically by weakening in different ways different characterizations of the Shapley value. Semivalues result by dropping efficiency, and include the Banzhaf and Shapley-Shubik indices as the most distinguished members. Thus, they can be seen as the family of decisiveness measures (sharing the properties shared by the two most popular indices, 'anonymity' among them) that depend on the structure of the game (i.e., the voting rule). This was already suggested by Weber, (1979) (see also Laruelle and Valenciano (2001b) and Carreras, Freixas and Puente (2002)). We have the following result[20]

Proposition 6 *All the three measures $\Phi_i(-,p)$, $\Phi_i^{i+}(-,p)$ and $\Phi_i^{i-}(-,p)$ $(i = 1, 2, .., n)$:*

(i) Become semivalues if and only if p is such that the probability of a configuration depends only on its size. Moreover, all regular semivalues are generated by $\Phi_i^{i+}(-,p)$ for p's in this family of probability distributions.

(ii) Become weak semivalues if for any two voters the probability of voting 'yes' is the same, i.e., $\gamma_i(p) = \gamma_j(p)$ for all i, j. And all weak semivalues are generated by $\Phi_i^{i+}(-,p)$ for p's in this family.

(iii) Are weighted weak semivalues, and the whole family of weighted weak semivalues is generated by $\Phi_i^{i+}(-,p)$ for different p's.

9 Comparison with other probabilistic models

Owen's (1975, 1988) multilinear extensions can be interpreted as a probabilistic model in which every voter independently from the others' behavior votes 'yes' with a certain probability. This particular class of probabilistic behaviors has been characterized in Proposition 1, and as has been pointed out is a particular case within the model considered here in which correlation is also admitted.

A comparison with Straffin's (1977, 1982, 1988) model as well as with Dubey, Neyman and Weber's (1981) extension to semivalues is interesting here. Straffin (1977, 1982, 1988)

[20]See Laruelle and Valenciano (2002) for a similar result on the more general domain of TU games.

proposes the following probabilistic model. Let $\mathfrak{N} = \{N_j\}_{j=1,2,..,m}$ be a partition of N into m disjoint subsets, and denote $M = \{1, 2, .., m\}$, and n_j the cardinal of N_j. Let $t = (t_1, .., t_m) \in [0, 1]^m$. Assume that for every $j = 1, 2, .., m$, every voter in N_j votes 'yes' with probability t_j and 'no' with probability $(1 - t_j)$. For every $S \subseteq N$, denote $S_j := S \cap N_j$, and s_j its cardinal. Then the probability of the configuration $S \subseteq N$, is

$$p_{(\mathfrak{N},t)}(S) = \prod_{j=1}^{m} t_j^{s_j} (1 - t_j)^{n_j - s_j}.$$

Now assume that each t_j is chosen independently from a probability distribution ξ_j on $[0, 1]$, and denote $\xi := (\xi_1, .., \xi_m)$. Straffin considers three special cases in which all ξ_k are the uniform distribution on $[0, 1]$, and, respectively, $m = 1$ ('homogeneity'); $m = n$ ('independence'); and $1 < m < n$ ('partial homogeneity').

In the most general case, i.e., for $1 \leq m \leq n$ and arbitrary probability measures ξ_k's, the probability of voter i 'affecting the outcome' of a decision by a voting rule W, if $i \in M_j$, according to Straffin is given by

$$Str_i(W, (\mathfrak{N}, \xi)) :=$$

$$\int_0^1 ... \int_0^1 \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} t_j^{s_j - 1} (1 - t_j)^{n_j - s_j} \prod_{k \in M \setminus j} t_k^{s_k} (1 - t_k)^{n_k - s_k} \; d\xi_1(t_1) ... d\xi_m(t_m). \tag{12}$$

As is well-known, under 'independence' this probability coincides with the Banzhaf index, while under 'homogeneity' coincides with the Shapley-Shubik index. The point is this: which is the relationship between Straffin's model and the one considered here? Is Straffin's more or less general? The answer is given by the following proposition that establishes the relation between (4) and Straffin's most general formula (12). Let $p_{(\mathfrak{N},\xi)}$ denote the resulting probability distribution over vote configurations in the general case specified by a partition $\mathfrak{N}$ and an N-vector of probability distributions ξ, that is,

$$p_{(\mathfrak{N},\xi)}(S) := \int_0^1 ... \int_0^1 \prod_{j=1}^{m} t_j^{s_j} (1 - t_j)^{n_j - s_j} \; d\xi_1(t_1) ... d\xi_m(t_m). \tag{13}$$

Then we have the following result:

Proposition 7 *For any partition $\mathfrak{N} = \{N_j\}_{j \in M}$ of N, and any M-vector of probability measures $\xi = (\xi_1, .., \xi_m)$ over $[0, 1]$,*

$$Str_i(W, (\mathfrak{N}, \xi)) = \int_0^1 ... \int_0^1 \Phi_i(W, p_{(\mathfrak{N},t)}) \; d\xi_1(t_1) ... d\xi_m(t_m) = \Phi_i(W, p_{(\mathfrak{N},\xi)}).$$

Proof. Assume if $i \in M_j$, then observe that in formula (12),

$$t_j^{s_j-1}(1-t_j)^{n_j-s_j} \prod_{k \in M \backslash j} t_k^{s_k}(1-t_k)^{n_k-s_k}$$

is the probability of the event: all voters in $S \backslash i$ vote 'yes' and all in $N \backslash S$ vote 'no', if for every $k = 1, 2, .., m$, every voter in N_k votes 'yes' with probability t_k and 'no' with probability $(1 - t_k)$. But mind that if S is winning and $S \backslash i$ is not, i would be decisive whatever her vote. In other words,

$$\sum_{\substack{S: i \in S \in W \\ S \backslash i \notin W}} t_j^{s_j-1}(1-t_j)^{n_j-s_j} \prod_{k \in M \backslash j} t_k^{s_k}(1-t_k)^{n_k-s_k} = \Phi_i(W, p_{(\mathfrak{N},t)}),$$

and the first equality is proved. Now by permuting addition an integration in (12), and taking into account (13) and (5), we obtain,

$$Str_i(W, (\mathfrak{N}, \xi)) =$$

$$\sum_{\substack{S: i \in S \in W \\ S \backslash i \notin W}} \int_0^1 ... \int_0^1 t_j^{s_j-1}(1-t_j)^{n_j-s_j} \prod_{k \in M \backslash j} t_k^{s_k}(1-t_k)^{n_k-s_k} d\xi_1(t_1)...d\xi_m(t_m)$$

$$= \sum_{\substack{S: i \in S \in W \\ S \backslash i \notin W}} (p_{(\mathfrak{N},\xi)}(S) + p_{(\mathfrak{N},\xi)}(S \backslash i)) = \Phi_i(W, p_{(\mathfrak{N},\xi)}).$$

And the proof is complete. ■

Thus we have the following conclusion: in strict terms the model considered here is *simpler and more general than Straffin's.* It is *simpler* for its formulation requires only the elementary notion of probability distribution over a finite set of events (i.e., a discrete random variable: the voting configuration), while Straffin's model is more complicated for it involves a 'double randomization,' that is, a (non discrete) distribution of probability over distributions of probability. And our model is *more general* in a precise sense from the previous proposition: whatever the probability distributions ξ_j's, Straffin model generates a probability distribution over vote configurations. That is, it provides a way of putting something (i.e., $p_{(\mathfrak{N},\xi)}$) within our black box p. But the reciprocal is *not* true: not all voting behaviors considered in our model can be generated from Straffin's (i.e., from (13))[21].

Dubey, Neyman and Weber's (1981) extend Straffin's homogeneity result to all semivalues, and Einy (1987) proves it holds also on the domain of simple games. They (and Einy for simple games) show that all semivalues emerge from formula (12), in the case $m = 1$

[21] For instance, let $n = 3$, and p such that $p(\{1, 2\}) = p(\{1, 3\}) = \frac{1}{2}$, and $p(S) = 0$, otherwise. This behavior (a 'boss' that controls the agenda and half the times has the support of one of two voters always holding opposite views, and half the times that of the other) cannot be generated from Straffin's model.

and for different probability measures $\xi(t)$ on $[0,1]$. More precisely there is a one-to-one correspondence between the set of semivalues and the set of probability measures on $[0,1]$. Compare the simplicity of the probabilistic model provided by $\Phi_i^{i+}(W,p)$ $(i=1,2,..,n)$, that according to Proposition 6-(i) generates all semivalues, and the unnecessary sophistication of the alluded particularization of (12).

10 Concluding remarks

The simple model presented in this paper provides a common basis to reinterpret power indices as well as some game theoretic 'cooperative solutions' that can be seen as extensions of this notion. We have deliberately avoided as much as possible the terms 'power' or 'voting power', and use preferentially the more neutral and precise 'decisiveness' and 'success' to avoid any argument about the use of words, and also to emphasize the relevance of both notions in connection with the voters' role in voting situations. The results of this reexamination in the light of this model can be summarized like this:

1. From the unifying point of view provided by this probabilistic model some power indices *but not all*, namely, Banzhaf's, Coleman's, Rae's and König-Bräuninger's indices, once adequately reformulated and generalized, have a precise interpretation as probabilistic measures of decisiveness or success under different conditional constraints. All these indices can be jointly justified as assessments *of the voting rule itself on the same normative grounds*, as based on the same probability distribution that assigns the same probability to all vote configuration. But mind that there is no conflict among these indices, for they all are based on the same model, they just measure different features. Banzhaf's seems the most preeminent, but those of Coleman deserve more attention than has usually been paid to them[22].

2. The same framework that supports the previous claim makes clear the lack of grounds to attach any *positive* or *descriptive* value to assess actual voting situations to any of these power indices (apart from the case of absolute ignorance about the voters and the context beyond the rule itself). But, on the other hand, this framework suggests a natural conceptual extension of these power indices to positive or descriptive assessments when this particular probability distribution is replaced by the one that best fits the specific real-world situation at hand, as well as to other normative-oriented measures if additional information were considered adequate to be reflected in the probability distribution.

3. Some other power indices (Shapley-Shubik, non normalized Holler-Packel) as well

[22]The importance given by the negotiators in Nice 2000 to the capacity of blocking (see Galloway, 2001) seems to corroborate this claim. The problem of forthcoming enlargements is attracting attention to Coleman's 'power of a collectivity to act'.

as some game-theoretic extensions (Owen's coalitional extension of Banzhaf index, as well as semivalues and weighted weak semivalues) fit into the model (only partially in some cases), but for probability distributions difficult to justify. Other power indices (Deagan-Packel, Johnston) cannot be accommodated within the model in no way. In either case this seems to corroborate the lack of clear normative arguments in support of any of them as measures of decisiveness. The possible positive/descriptive meaning in a completely different sense of any of these notions in certain situations is not discussed here, for it is beyond the objectives of this work.

4. This raises the question of alternative meanings of the terms 'power' or 'voting power', to which the latter indices seem to refer on insufficiently clear grounds. There is Felsenthal and Machover's (1998) obscure notion of 'P-power' associated to a situation in which the main ingredient of a voting situation seems to be the distribution of some 'spoils'. But as far as we know there is no coherent general formulation of this notion yet. So far the term 'P-power' only covers an insufficiently specified notion, although it possibly points out to a real 'hole' in the theory.

5. Points 1 and 2 do not mean that power indices exhaust what is to be said about voting situations. Not in the least. There is much more to say from a positive point of view about real world voting situations than what power indices or their positive extensions may say. After all, success and decisiveness notions refer to the formal role played by voters in voting situations. That this is not all that is to be said about voting situations is obvious, and is corroborated by the abundant criticisms of power indices, however ill-founded these criticisms may often be. The proliferation of alternative models has evidently to do with the insufficiencies of power indices. It is worth remarking the absence of any explicit genuine game theoretic ingredient in the whole approach developed here. Of course, any real world voting situation involves rational interaction of the voters, which interests game theory. But in this model this game-like background is only implicit within the black box summarizing the voters behavior. There seems to be still much to be said about voting situations from a genuine game theoretic point of view, beyond what power indices can tell.

6. In comparison with other probabilistic models of the voters' behavior the one considered here seems at once simpler and more general. Our model includes Straffin's as a particular way among others of filling the voters' behavior's black box, and it is definitely simpler. The sophistication of Straffin's model with its double randomization has caused a great fascination over social scientists, for it provides a suggestive model which in two 'extreme' particular cases yields the two most popular power indices, Shapley-Shubik's (homogeneity case) and Banzhaf's (independence case), and allows for 'tailored power

indices' combining features of both (partial homogeneity case) (Weber 1988, p. 78). But in our view it offers a false way of eclectically escaping the unanswered criticisms about the Shapley-Shubik index by Banzhaf (1965) and Coleman (1986). The simpler model discussed here makes evident instead the different normative worth of both indices as measures of decisiveness, and provides a very simple framework in which a variety of notions can be coherently integrated.

7. In two previous papers we revised the axiomatic foundations of the Shapley-Shubik index and the Banzhaf index, and of the semivalues (Laruelle and Valenciano, 2001a, 2000, 2001b). Perhaps we have done our way in the wrong order, starting with the axiomatic foundations and only then reexamining the probabilistic nature of the concepts involved. As a conclusion of this tour we fully agree with Straffin's (1988) words: 'I believe that it [the axiomatic approach] is less effective than the probability approach in giving clear heuristic advice about which power index is applicable to which voting situation.'

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