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A B S T R A C T

This paper deals with the assessment of inequality in the distribution of voting power. As voting procedures are modeled as simple games and power evaluated through power indices, two approaches are possible to deal with inequality in this context, depending on whether the power pro¯les generated by some power index or the simple games that model the voting rules are taken as primitives. In both cases the mechanical application of previous results does not make sense. This paper uses the ¯rst approach to found axiomatically some inequality indices in this speci¯c context and discusses some di±culties with the second approach.

KEYWORDS: Inequality; Power Indices; Voting Power; Collective Decision-Making.

1 INTRODUCTION

This paper is concerned with the assessment of inequality in the distribution of power in collective decision-making procedures. This issue arises naturally in di®erent contexts as a matter of practical interest. For instance, in the comparison between alternative speci¯-cations of voting rules for decision-making by a given set of agents (councils, committees, parliaments, etc.). A precise tool to make such comparisons would be of great interest for the design of voting procedures. A relevant case-study, the evolution through years of the distribution of power among the citizens in the European Union has in fact been the original motivation of this work (Laruelle, 1998).

With this practical-design motivation in mind, the aim of this paper is to provide a tool to measure inequality in voting procedures. Such a tool should be axiomatically grounded -it may be convenient to remark- in the speci¯c context of voting power. Usually decision-making procedures are formally described as simple games and power evaluated through power indices. Thus, two approaches are possible to deal with inequality in this context, depending on whether the power pro¯les generated by some power index or the simple games that model the voting rules are taken es primitives.

The ¯rst option immediately suggests to apply some of the indices provided by the rich literature on inequality. But this literature is concerned with the distribution of income (see, e.g., Atkinson (1970), Kolm (1976), Shorrocks (1980), Weymark (1981), Yaari (1988) and Porath and Gilboa (1994)), while here we are concerned with a completely di®erent and more elusive concept: "power". Moreover, most axioms in this literature not even ¯t a domain consisting of a ¯nite set of pro¯les as it happens to be the case. Therefore the adequacy of tools developed in a thoroughly di®erent conceptual context is not obvious and would require at least a re-foundation. This is the approach, outlined a few lines below, adopted here.

Alternatively, the fact that simple games can be used to model voting procedures may suggest adopting Einy and Peleg's (1991) approach to deal with inequality in TU-games. They directly axiomatize a family of inequality measures, which are generalized Gini functions of the Shapley value of the games, using these games as primitives. According to them, any similar endeavour taking as primitives the outcomes of any particular solution concept would yield an ad hoc measure of inequality, because of the multiplicity of solutions. But, paradoxically, the outcome of their approach is that only one solution, the Shapley value, ¯lters through their axioms. In fact, they implicitly assume e±ciency in the underlying solutions which allows them to exclude from consideration any semivalue but the Shapley value. More generally, their approach has the drawback of mixing the assessment of two di®erent things: "value" and "inequality" in the distribution of it. As a consequence, some of their axioms are not transparent. As to the restriction of their results to simple games as models of voting procedures, it is even more arguable. In fact, Einy and Peleg consider this application one of the motivations of their work, but again the specificity of the context poses some problems. First, the implicit assumption of efficiency is especially arguable in the context of simple games as models of decision-making procedures. Indeed efficiency is not any more a natural requirement for a solution in this context: 1, the worth of the grand coalition, cannot be interpreted as a cake that has to be (efficiently) distributed among the players[1] (Laruelle and Valenciano, 1999). Moreover, this assumption results in arbitrarily considering the Banzhaf index, one of the most widely applied power indices and arguably the most suitable in many applications (see Felsenthal and Machover's (1998) and Laruelle (1999)), as any other semivalue, as "unreasonable". Second, the addition of two simple games is not a simple game. Therefore those of their axioms that use addition of games (namely, "restricted additivity" and "independence") should be reformulated in this specific context. Although the reformulation is possible (in Section 4 we sketch this adaptation and compare their results with ours), the natural translation of some of their axioms lacks a compelling justification in terms of voting procedures. In the concluding remarks, we come back to this point and show how one of their axioms even seems quite counterintuitive in this specific context. On the other hand, they do not single out an index, but a family, and only consider a fixed number of players. Therefore any possible application of their work would require some further choice to single out an index. But they give no hint for this further choice and the comparison of inequality in games with different number of players is not considered.

In sum, the mechanical application of indices axiomatically grounded on either approach would not be justified. In both cases a re-foundation is previously necessary. This paper is a first step in this direction. In a first step it seems only prudent to separate power and the inequality in the distribution of it, necessarily intermingled in Einy and Peleg's approach. Consequently we take the first approach. Nevertheless, we give some clues and point out some problems for the adaptation of Einy and Peleg's approach and show the relation of our indices with theirs.

Thus, our approach can be summarized as follows. Voting rules are modeled as simple superadditive games. The distribution of power among the agents is then evaluated by power indices that associate a power profile with each game. To compare power profiles according to the degree of inequality in the distribution of power some indices (i.e., real-valued functions on the set of all possible power profiles) are axiomatically characterized.

[1] At least if power is interpreted in the sense we use the term here, that is, the a priori capacity to influence the outcome of a vote ("I-power" in Felsenthal and Machover's (1998) terminology).

To this end we propose properties for an inequality index that have a meaning in terms of the involved concept of power. As power and inequality are here explicitly separated, di®erent power indices can be considered. So far, no agreement has been reached among the scholars concerning the choice of the most suitable index. In fact, it may depend on the particular context (Laruelle, 1999). Therefore in this paper, we deal with the two best known power indices, that is to say, the Shapley-Shubik index (1954) and the Banzhaf index (1965), but in fact our treatment of the latter could be easily extended to any semivalue. As the set of Shapley-Shubik power pro¯les di®ers from the set of Banzhaf power pro¯les, they are separately dealt with.

As the number of n-person simple games is ¯nite, the number of possible power pro¯les, is ¯nite too whatever the power index being used. In order to introduce and formalize in a more tractable way any index of inequality in this context, we extend the set of feasible pro¯les by convexifying these ¯nite sets. This convexi¯cation corresponds to enlarging the underlying set of games to the set of all convex combinations of simple superadditive games. Any game from this set can be interpreted as a lottery on simple games, in which the worth of a coalition is its probability of being winning. Then, consistently, the Shapley value and the Banzhaf semivalue of this game can be interpreted as an expected-power pro¯le. In this paper, we therefore model and rank decision-making processes and lotteries on them. As a result, the domain of our inequality indices are closed convex sets of power pro¯les, instead the usual R^n_+ for income pro¯les. Moreover, this underlying choice gives support to a solid assumption in this context: our restricted or not "expected inequality on co-ranked pro¯les."

The results obtained in this paper are the following. In the case of Shapley-Shubik power pro¯les, two plausible properties restrict drastically the class of indices to a family closely related to Einy and Peleg's family. Then, adding some conditions we characterize, up to a positive constant, a unique inequality index for any ¯xed number of players. By adding any of two alternative equivalence principles we extend in two ways this index to deal with comparisons of power pro¯les with di®erent number of players. In the case of Banzhaf power pro¯les a distinction is also made between absolute and relative inequality indices, so that two indices, one of either class, are characterized up to a positive constant for any ¯xed number of players. Four inequality indices arise then to deal with comparisons of power pro¯les with di®erent number of players. The working of the indices is then illustrated in the UN Security Council.

The paper is organized as follows: in Section 2 the basic game theoretical background is given. In Section 3 the class of simple superadditive games is extended to deal with lotteries on decision-making processes. Then Dubey and Shapley's axiomatizations of

Shapley-Shubik and Banzhaf indices are extended to this wider domain. In Section 4, an inequality index to compare Shapley-Shubik power profiles is axiomatically characterized for a fixed number of agents, and then extended in two ways to compare profiles with different number of agents. In Section 5 a similar construction is done for Banzhaf power profiles, where the distinction between relative and absolute inequality give rise to two couples of indices. In Section 6 the study of the inequality in the UN Security Council questions the 1965 reform of its decision-procedure. Finally, Section 7 concludes with some critical remarks on the results presented in this paper and a brief discussion on some lines for further research.

2 BASIC GAME THEORETICAL BACKGROUND

A cooperative transferable utility (TU) game is a pair $(N; v)$, where $N = \{1; \ldots; n\}$ denotes the set of players and v a function which assigns a real number to each non-empty subset or coalition of N and $v(\emptyset) = 0$. The number of players in a coalition S is denoted s. In a $(0; 1)$-game, the function v only assigns the values 0 and 1. In these games the coalitions with worth 1 are referred to as winning, while those with worth 0 as losing. A player i is said to be a swinger in a coalition S, if S is winning and $S \setminus \{i\}$ is not. For any coalition $S \subseteq N$, the S-unanimity game, denoted $(N; u^S)$, is the game such that

$$u^S(T) = \begin{cases} 1 & \text{if } T \supseteq S \\ 0 & \text{otherwise.} \end{cases}$$

A simple game is a $(0; 1)$-game that is not identically 0 and obeys the condition of monotonicity: $v(T) \geq v(S)$ whenever $T \supseteq S$. A game is superadditive if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. In the context of simple games, the superadditivity property is equivalent to the condition: $v(S) + v(N \setminus S) \leq 1$ for all $S \subset N$. Let SG_n (resp., G_n) denote the set of all simple (resp., the set of all) superadditive n-person games. Note that G_n is included in the $2^n - 1$ euclidean space.

A decision-making procedure can be modeled as a $(0; 1)$-game where the winning coalitions are defined as those which can make a decision without the vote of the remaining players. In this context we usually have that (i) the unanimity of the players can make a decision; (ii) any subset of a losing coalition is losing; and (iii), any two nonintersecting coalitions cannot be winning at the same time. Thus any decision-making process satisfying these conditions can be modeled by a simple superadditive game.

A power index is a function $\Phi : SG_n \to \mathbb{R}^n$ that assigns to each simple superadditive game $(N; v)$ a vector or power profile $\Phi(v)$ whose ith component is interpreted as a measure of the influence that player i can exert on the outcome. To evaluate the distribution of power among the players, the two best known power indices are the Shapley-Shubik (1954) index and the Banzhaf (1965) index. Formally, the Shapley-Shubik index is given by $Sh_n(v) = (\varphi_1(v), \dots, \varphi_n(v))$, where

$$\varphi_i(v) = \sum_{\substack{S \subseteq N \\ (S \ni i)}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus \{i\})\right], \quad i = 1, \dots, n. \tag{1}$$

While the Banzhaf index is given by $Bz_n(v) = (\beta_1(v), \dots, \beta_n(v))$, where

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{\substack{S \subseteq N \\ (S \ni i)}} \left[v(S) - v(S \setminus \{i\})\right], \quad i = 1, \dots, n. \tag{2}$$

Both $\varphi_i(v)$ and $\beta_i(v)$ can be interpreted as the probability of player i being a swinger in the coalition voting a proposal according to the voting rule modeled by v. They differ in the expectations about this coalition. The first index corresponds to the assumption that S is formed by i and the players who precede her or him in an ordering chosen at random. While $\beta_i(v)$ in coalition S is chosen at random among all coalitions to which i belongs. Both indices are in fact the restriction to SG_n of two well-known linear maps from G_n to R^n, the Shapley value and the Banzhaf semivalue, that we will denote Sh_n and Bz_n too.

3 LOTTERIES ON VOTING PROCEDURES

We are concerned with the problem of ranking decision-making procedures, taking power profiles generated either by the Shapley-Shubik index or by the Banzhaf index as primitives. Note that in both cases the number of possible power profiles, that is, the sets $Sh_n(SG_n)$ and $Bz_n(SG_n)$, are finite, as the number of simple superadditive n-person games is. In order to introduce and formalize in a more tractable way any index of inequality in this context, it is more convenient to extend the set of feasible profiles by convexifying these finite sets. That is, considering the convex hull of the set of the Shapley-Shubik (resp., Banzhaf) profiles of all n-person simple superadditive games as the set of profiles to deal with. Let us denote this set $Co(Sh_n(SG_n))$ (resp., $Co(Bz_n(SG_n))$).

This choice makes sense. The Shapley value, Sh_n, and the Banzhaf semivalue, Bz_n, both defined on G_n, are linear maps. Therefore, $Co(Sh_n(SG_n) = Sh_n(Co(SG_n))$, where the last set is formed by the Shapley values of the convex hull of the set of all n-person simple superadditive games. Similarly, with the same notation, we have $Co(Bz_n(SG_n)) = Bz_n(Co(SG_n))$. So, in both cases this convexification corresponds to enlarging the underlying set of games to $Co(SG_n)$, the set of all convex combinations of simple superadditive games. Games in this set can be interpreted as -and identified with- lotteries on simple superadditive games if the worth of a coalition in a lottery is defined as the expected worth in the involved games, that is, its probability of being winning. Then, consistently, the power profile of a lottery on simple superadditive games, interpretable as an expected-power profile, is the value of the corresponding convex combination of games, given by formulae (1) and (2) that, as commented above, make sense for any game in G_n and in $Co(SG_n)$ in particular.

Moreover, an axiomatic characterization of both indices in $Co(SG_n)$ can be easily achieved using some of the following assumptions concerning a map: $\Phi : Co(SG_n) \to R^n$; $v \to \Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$. Some axioms that are common requirements both in SG_n and G_n also make sense in this domain. These are:

Anonymity (AN): For any permutation π of N, and any $i \in N$, $\Phi_i(\pi v) = \Phi_{\pi(i)}(v)$; where $(\pi v)(S) := v(\pi(S))$.

Null Player (NP): If $v(S) = v(S \setminus \{i\})$ for all S; then $\Phi_i(v) = 0$.

The anonymity axiom states that a player's measure of power does not depend on her or his name. The null player axiom postulates that if a player's presence in any coalition does not contribute to increase its probability of being winning, this player has no power.

The following two axioms, that distinguish the Shapley-Shubik and the Banzhaf indices in SG_n, also make sense in this domain. They are:

Constant Total Power (CTP): $\sum_{i=1} \Phi_i(v) = 1$.

Banzhaf Total Power (BTP): $\sum_{i=1} \Phi_i(v) = \frac{1}{2^{n-1}} \mu(v)$; where $\mu(v) = \sum_{i=1} \eta_i(v)$ and $\eta_i(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} (v(S) - v(S \setminus \{i\}))$.

The constant total power axiom requires that all players' measures of power add up to 1 in any game. The Banzhaf total power axiom states that the players' measures of power add up to the expected total number of swings divided by the number of coalitions to which any player belongs.

In SG_n these axioms together with Dubey's (1975) "transfer" axiom permit characterizing both indices. In fact, the transfer axiom, devoid of any compelling interpretation, plays in SG_n the role that linearity plays in G_n. Moreover, the transfer axiom does not make sense in $Co(SG_n)$, not even mathematically, for it is not closed with respect to the operators "$\vee$" and "$\wedge$". The same can be said with respect to linearity. Instead, the right assumption, both from the mathematical and the intuitive point of view, in our intermediate domain is:

Expected Power (EP): For all $v, w \in Co(SG_n)$; and $\lambda \in [0,1]$; $\Phi(\lambda v + (1-\lambda)w) = \lambda\Phi(v) + (1-\lambda)\Phi(w)$.

The meaning of this axiom is clear: it states that a player's measure of power in a lottery is the expected power in the involved games. This assumption is especially natural if power, as measured by both indices, is interpreted as an expectation. Then we can easily extend Dubey and Shapley's (1979) characterization to $Co(SG_n)$:

Theorem 1 Let $\Phi : Co(SG_n) \to \mathbb{R}^n$; be an index of power.

1. The only Φ that satisfies anonymity, null player, expected power and constant total power is the Shapley value.

2. The only Φ that satisfies anonymity, null player, expected power and Banzhaf total power is the Banzhaf semivalue.

Proof: The proof is a simple adaptation in the set $Co(SG_n)$ of Dubey and Shapley's characterization of the Shapley-Shubik index and of the Banzhaf index in SG_n.
(i) The Shapley value obviously satisfies all four axioms. With regard to uniqueness, first note that EP on Co(SGn) implies Dubey and Shapley's transfer axiom on SG_n. Moreover, NP, AN and CTP restricted to SG_n yield Dubey and Shapley's other axioms. Therefore the restriction of Φ to SG_n is the Shapley value. Then, by EP, Φ and the Shapley value must also coincide in all Co(SGn).
(ii) Similarly, the Banzhaf semivalue is the only value that satisfies AN, NP, EP and BTP. ■

The difference between the Shapley value and the Banzhaf semivalue lies in one axiom: the constant or the Banzhaf total power. The first axiom, usually referred to as "efficiency" in spite of the lack of meaning of this term in the context of value as a measure of power (Laruelle and Valenciano, 1999), entails a constant addition of the players' power. While for the Banzhaf semivalue, the (variable) addition of all players' power can be considered as a measure of the expected ease of making a decision. Dubey and Shapley interpret $\mu(v)$ in the context of decision-making processes as "a kind of democratic participation index, measuring the decision's rule sensitivity to the desires of the 'average voter' or to the 'public will'." (Dubey and Shapley, (1979), p. 106). The same interpretation can be given in the context of lotteries on voting rules in terms of expectations.

According to the above discussion, even if $Co(SG_n)$ is the common starting point, we have two different sets of power profiles to deal with depending on the index used to generate them. If the Shapley index is used, this set is $Co(Sh_n(SG_n))$, that is, the $(n-1)$-dimensional simplex whose extreme points are the vectors of the natural basis of $\mathbb{R}^n$. We will denote Δ_n this simplex. If Banzhaf is the index being used, this set is $Co(Bz_n(SG_n))$, a symmetric (i.e., closed under permutations of the players), compact and convex subset of $\mathbb{R}^n_+$.

4 INEQUALITY INDICES FOR SHAPLEY-SHUBIK POWER PROFILES

If Shapley-Shubik is the index used to generate the power profiles, we have the following framework for each number n of players:

$$Co(SG_n) \xrightarrow{Sh_n} \Delta_n \xrightarrow{I_n} \mathbb{R}.$$

That is, an inequality index is a function that associates a number with each power profile in the $(n-1)$-simplex Δ_n that is used to rank power profiles according to the so assessed degree of inequality. In fact, in this way we have a composite index $I_n \circ Sh_n$ that ranks games in $Co(SG_n)$ and in SG_n in particular.

As stated before, the choice of an inequality index I_n should be based on the properties one desires the index to satisfy, and these properties must be consistent with the properties of the power index being used, Shapley-Shubik in this case. As recalled in the previous section, the Shapley-Shubik value is characterized as the unique power index in $Co(SG_n)$ that satisfies anonymity, null player, expected power and constant total power. Constant total power is behind the choice of the domain for I_n, that is, the $(n-1)$-simplex Δ_n. Consistent with the anonymity axiom of Sh_n, it is natural to require the following condition that we will refer to as anonymity too:

Anonymity (AN): For all $\varphi \in \Delta_n$, and any permutation π of N: $I_n(\varphi_1, ..., \varphi_n) = I_n(\varphi_{\pi(1)}, ..., \varphi_{\pi(n)})$.

The meaning of this axiom, usual in other contexts, is obvious: the degree of inequality in a power profile does not depend on how are labelled its components.

Now we turn our attention to the expected power axiom characterizing Sh_n. In fact this is equivalent to require the convex linearity of the power index, a weak form of linearity restricted to convex combinations. If I_n satisfied convex linearity, this would compose nicely with Sh_n's convex linearity, and would permit to interchange I_n and randomization. But it is clear that asking for linearity unrestrictedly for an inequality index in Δ_n would not work. This condition together with anonymity would yield a constant index because any point in the simplex Δ_n is a convex combination of its extreme points. So, for any profile $\varphi \in \Delta_n$, we would have: $I_n(\varphi_1, \ldots, \varphi_n) = \varphi_1 I_n(1, 0, \ldots, 0) + \ldots + \varphi_n I_n(0, \ldots, 0, 1) = I_n(1, 0, \ldots, 0)$. In fact, this is the effect of requiring, together with anonymity, convex linearity on profiles in which the players are differently ranked according to their power. Convex linearity can only be required on "co-ranked" profiles, that is, pairs of profiles φ, φ' such that for all $i, j \in N$, $\varphi_i < \varphi_j \Rightarrow \varphi'_i \leq \varphi'_j$. Moreover, taking into account the interpretation of convex combinations as random mixtures, this requirement would mean that lotteries on voting procedures, in which the players are equally ranked according to their power, are ranked according to a von Neumann-Morgenstern preference ordering. In other words, the inequality in a lottery on decision-making processes (with identically ranked players) is the expected inequality of the involved decision-making processes. This seems a very reasonable assumption in any context in which, as in this case, ordering lotteries on a given fixed set of alternatives is the point at issue. So we propose the following condition:

Expected Inequality on Co-ranked profiles (EIC): For all pair of co-ranked power profiles $\varphi, \varphi' \in \Delta_n$, and all $\lambda \in [0, 1]$: $I_n(\lambda\varphi + (1 - \lambda)\varphi') = \lambda I_n(\varphi) + (1 - \lambda) I_n(\varphi')$.

Note that any index satisfying anonymity is fully determined by its restriction to any of the $n!$ sets of co-ranked vectors in Δ_n. The following lemma shows that any of these sets is an $(n - 1)$-subsimplex of the simplex Δ_n, and is the convex-hull of the Shapley-Shubik power profiles of n unanimity games. Using e^k to denote the vector where the k first components are equal to $1/k$ and the others are null, we have:

Lemma 1 The set of all power profiles $(\varphi_1, \ldots, \varphi_n) \in \Delta_n$ such that $\varphi_1 \geq \ldots \geq \varphi_n$ is an $(n - 1)$-simplex whose extreme points are: e^1, ...,e^k,..., e^n. Moreover, taking $\varphi_{n+1} = 0$, we have:

$$(\varphi_1, \ldots, \varphi_n) = \sum_{k=1}^{n} k(\varphi_k - \varphi_{k+1}) e^k.$$

Proof: It suffices to check that $(\varphi_1, \ldots, \varphi_n)$ can be uniquely written as a convex combination of e^1, ...,e^k,..., e^n to get the result.

■

Just permuting the components we get the extreme points of the other $n! - 1$ simplices of co-ranked profiles. In fact, they form a simplicial partition of Δ_n. This means that any power profile in Δ_n can be uniquely expressed as a convex combination of the Shapley-Shubik power profiles of n unanimity games co-ranked with it. In sum, any index satisfying expected inequality on co-ranked profiles would rank co-ranked power profiles according to a preference ordering satisfying von Neumann-Morgenstern assumptions, and would be fully determined by the values of the index for these $2^n - 1$ profiles. If the index also satisfies anonymity then it would be fully determined by $I_n(e^1), \ldots, I_n(e^k), \ldots, I_n(e^n)$. More precisely, we have the following result:

Theorem 2 An index $I_n : \Delta_n \to \mathbb{R}$, satisfies anonymity and expected inequality on co-ranked profiles if and only if it can be written as:

$$I_n(\varphi_1, \ldots, \varphi_n) = \sum_{k=1} \left(k I_n(e^k) - (k-1) I_n(e^{k-1}) \right) \hat{\varphi}_k, \tag{3}$$

where $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n)$ denotes the vector that results by re-ordering φ's components decreasingly, and $I_n(e^0)$ is set up equal to 0.

Proof: First it is easy to check that the index given by (3) satisfies AN and EIC. Now let I_n be an index satisfying these axioms. By AN, $I_n(\varphi) = I_n(\hat{\varphi})$. By Lemma 1 and EIC, we obtain:

$$\begin{aligned} I_n(\hat{\varphi}_1, \ldots, \hat{\varphi}_n) &= I_n\left(\sum_{k=1} k(\hat{\varphi}_k - \hat{\varphi}_{k+1}) e^k \right) \\ &= \sum_{k=1} k(\hat{\varphi}_k - \hat{\varphi}_{k+1}) I_n(e^k) \\ &= \sum_{k=1} \left(k I_n(e^k) - (k-1) I_n(e^{k-1}) \right) \hat{\varphi}_k. \end{aligned}$$

■

So, these two conditions, anonymity and expected inequality on co-ranked profiles, restrict drastically the class of indices. In fact, this is Einy and Peleg's first family of indices (Theorem 3.1). More precisely, comparing (3) and formula (3.4) in Einy and Peleg (1991), it easily follows the following

Corollary 1 An ordering on $Co(SG_n)$ is the restriction to this domain of an ordering on G_n that satisfies the assumptions in Theorem 3.1 of Einy and Peleg (1991) if and only if it is representable by a composite index $I_n \circ Sh_n$ where I_n satisfies anonymity and expected inequality on co-ranked profiles.

This family of orderings/indices on $Co(SG_n)$ can be characterized also directly adapting Einy and Peleg's axioms to our domain. This can be done by means of some plausible adaptations (for instance, using convex combinations instead of additions of games, and taking into account that the only inessential games in our domain are the convex combinations of dictatorships). But the result, though mathematically correct, is not completely satisfactory. As we discuss with more detail in the concluding remarks, the natural adaptation of some of their axioms lacks intuitive appeal in the context of voting procedures.

Observe also that formula (3) is more expressive than formula (3.4) of Einy and Peleg (1991), for it gives a precise meaning to the coefficients about which Einy and Peleg's formula says nothing. In particular it permits at least a plausible further narrowing of the family, as we presently show. As we have mentioned in the introduction, the original motivation of our work was to assess inequality in the distribution of power in real world collective decision-making situations. This requires an index, not just a family of them. So, accepting anonymity and expected inequality on co-ranked profiles, a further narrowing of the resulting class of indices is still to be done. In order to single out an index, according to formula (3), a choice for the values of $I_n(e^k)$ $(k = 1, \dots, n)$ is necessary (and sufficient).

Some reasonable constraints on this choice can easily be made. The comparison of the degree of inequality in profiles in which the power is equally shared by a group of players is obvious: the bigger the number of null players the bigger the degree of inequality. For each nonempty $S \subseteq N$, let e^S denote the profile whose S-components are $\frac{1}{s}$ and the rest are 0. Then, a plausible requirement is:

$$I_n\left(e^S\right) > I_n\left(e^T\right) \quad \text{whenever } s < t.$$

A further natural condition is to require some relative-to-size sensitivity to the addition of null players, that is, for all $S, T \subset N$ such that $s \leq t$ to require

$$I_n\left(e^S\right) - I_n\left(e^{S \cup \{i\}}\right) \geq I_n\left(e^T\right) - I_n\left(e^{T \cup \{j\}}\right)$$

for all $i \in N \setminus S$, $j \in N \setminus T$. The first condition is a form of monotonicity, while the second is a form of convexity. Both restrict the range of choice of $I_n(e^k)$, and therefore the coefficients in formula (3). In fact, adding these conditions entail, respectively, the positivity and the nondecreasing order of the coefficients in (3), exactly the two further conditions Einy and Peleg (Theorems 3.4 and 3.6) get for their coefficients by adding their "monotonicity" and "equality mindedness" requirements (see also Weymark (1981) and Yaari (1988)). But none of this assumptions is strong enough to single out an index. The same can be said about other assumptions common in the literature of inequality, as for instance the "progressive transfer".

Thus, in this point any step beyond is arguable, though necessary to specify an index. Our choice here is the simplest one compatible with the above conditions: we just require that these differences are constant and positive. As we will show it yields a tractable index. We have then the following condition:

Constant Sensitivity to Null Players (CSNP): There exists a constant $K_n > 0$ such that for all $S \subset N$, and all $i \in N \setminus S$, $I_n\left(e^S\right) - I_n\left(e^{S \cup \{i\}}\right) = K_n$.

It seems clear that the power profile in which the power is shared equally among all players corresponds to the minimum of inequality. On pure normalizing grounds we can assign to this power profile a zero index of inequality. That is:

Zero Normalization (ZN): $I_n\left(e^N\right) = 0$.

It can be shown that these axioms are not independent. Anonymity is implied by two of the other axioms as the following lemma shows:

Lemma 2 *If an index $I_n : \Delta_n \to \mathbb{R}$ satisfies expected inequality on co-ranked profiles and constant sensitivity to null players, then it satisfies anonymity.*

Proof: Let I_n be an index satisfying EIC and CSNP. Let $s < n$. Applying $(n - s)$ times CSNP, one easily obtains: $I_n\left(e^S\right) = I_n\left(e^N\right) + (n - s)K_n$, that is, $I_n\left(e^S\right)$ only depends on s: So we have $s = t \Rightarrow I_n\left(e^S\right) = I_n\left(e^T\right)$: Now let $\varphi \in \Delta_n$. By Lemma 1, φ can be uniquely written as a convex combination of the extreme points of an $(n - 1)$- simplex of power profiles co-ranked with it. If these extreme points are $e^{S_1}, e^{S_2}, ..., e^{S_n}$; where the cardinality of S_k is k, we have: $\varphi = \sum_{k=1}^{n} \lambda_k e^{S_k}$ (for some $\lambda_k \geq 0$ such that $\sum_{k=1}^{n} \lambda_k = 1$). Then by EIC: $I_n(\varphi) = \sum_{k=1}^{n} \lambda_k I_n(e^{S_k})$. But then $I_n(\varphi) = I_n(\pi\varphi)$ for any permutation π of N, for each $I_n(e^{S_k})$ only depends on k.

■

The remaining three axioms uniquely characterize (up to a constant) an inequality index as follows:

Theorem 3 *There is a unique (up to a positive proportionality constant K_n) inequality index $I_n : \Delta_n \to \mathbb{R}$, satisfying expected inequality on co-ranked profiles, constant sensitivity to null players and zero normalization, and it is given by*

$$I_n(\varphi_1, ..., \varphi_n) = K_n \sum_{k=1}^{n} (n - 2k + 1)\hat{\varphi}_k. \tag{4}$$

Proof: First, it is easy to check that the index given by (4) satisfies these axioms. Now let I_n be an index satisfying them. By Lemma 2, it satisfies AN too. Thus, by Theorem 2, $I_n(\varphi_1, ..., \varphi_n)$ is given by (3). Denoting $K_n := I_n\left(1^S\right) - I_n\left(1^{S\cup\{i\}}\right)$, constant for any pair i, S such that $i \notin S \subseteq N$ by CSNP, and using ZN, it follows easily that $I_n(e^k) = (n - k)K_n$. Then, substituting in (3), it yields (4).

■

In the preceding discussion the number of players has been considered fixed. But in certain cases one can be interested in comparing power profiles involving different number of players. For instance, in the case mentioned in the introduction -the evolution of the distribution of power in the European Union along the years- the number of countries and the number of citizens are not constant. Then an inequality index should be defined as a function $I : \bigcup_n \Delta_n \to \mathbb{R}$, while the above three axioms would only characterize up to a family of constants $(K_n)_{n \in \mathbb{N}}$, a family of indices $I = \{I_n : \Delta_n \to \mathbb{R} \mid n = 2, 3, \ldots\}$, one for each number of players.

In fact, the domain of each function in this family is different, and only zero normalization connects the value of the index for different number of players establishing a "common zero" for flat profiles. So, even if the above axioms are accepted for any number of players n, there is still a constant K_n undetermined for each number of players. The choice of this constant is immaterial for a fixed number of players. But this choice matters if power profiles with different number of players are to be compared by means of the corresponding I_n. In this case the above family of functions can be used to define a function $I : \bigcup_n \Delta_n \to \mathbb{R}$. Assuming the three axioms for any number of players, an index I would be completely specified if we postulate some equivalence principle relating the inequality index of profiles with different number of players. A weak reasonable principle would be requiring $I(1, 0_{n-1}) \leq I(1, 0_n)$, where $(1, 0_n) = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, that is, the degree of inequality cannot increase if we reduce the number of 0-players in a dictatorship. Using formula (4), we have $I(1, 0_{n-1}) = (n-1)K_n$, thus this would entail for the constants the condition $(n-1)K_n \leq nK_{n+1}$, for $n = 2, \ldots$ Within this range of choices we underline two that can be defended on their own grounds. A most simple choice is that of a common degree of inequality for any dictatorship, whatever the number of players. That is:

Dictator Player Equivalence Principle (DPEP): For all $n = 2, 3, \ldots$,

$$I(1, 0_{n-1}) = I(1, 0_n).$$

Note that for a given number of players, the dictatorship is the situation in which the degree of inequality is maximal. Therefore the above mentioned principle establishes a "common maximum" of inequality for any number of players, which is reached when there is a dictator. This entails for the constants the relation $K_n = (\frac{1}{n-1})K$, where K is an arbitrary positive constant.

This principle can be criticized on the basis that, from the inequality point of view, it can be argued that the bigger the number of players in a dictatorship the worse. A different equivalence principle that is sensitive to this idea is the following:

Null Players Equivalence Principle (NPEP): For all $n = 2, 3, \ldots$,

$$I(1, 0_{n-1}) = I(\tfrac{1}{2}, \tfrac{1}{2}, 0_{n-1}).$$

That is, the index of the profile associated with a dictatorship is the same as that of a profile in which the power of the dictator is equally split into that of two members, without changing the number of null players. This implies a simple relation between the constants: all of them are equal, that is, $K_n = K$, where K is an arbitrary positive constant. Note that this axiom is a weaker form of a general and clear principle that is satisfied by the index so characterized below, as it can be easily checked. The general principle considers as equivalent from the inequality point of view profiles with different number of players in which the power is equally shared by a group of them, as far as the number of null players is the same in both.

So, two indices (depending on which equivalence principle is assumed) are characterized up to a positive constant:

Theorem 4 There is a unique (up to a positive proportionality constant K) inequality index: $\bigcup_n \Phi_n \to \mathbb{R}$ satisfying expected inequality on co-ranked profiles, constant sensitivity to null players and zero normalization for any n, and satisfying the dictator player equivalence principle (respectively the null players equivalence principle). They are respectively given by:

$$I^{DP}(\varphi_1, ..., \varphi_n) = K\left(\frac{1}{n-1}\right) \sum_{k=1} (n - 2k + 1)\alpha_k, \quad (5)$$

$$I^{NP}(\varphi_1, ..., \varphi_n) = K \sum_{k=1} (n - 2k + 1)\alpha_k. \quad (6)$$

Reconsidering the constant sensitivity to null players, another characterization of I^{NP} can be given. Let us examine the effect of adding a null player to a decision-making process. It is easy to check that the following equation is satisfied:

$$I^{NP}(\varphi_1, ..., \varphi_n, 0) = I^{NP}(\varphi_1, ..., \varphi_n) + K. \quad (7)$$

This property could be properly called "constant sensitivity to null players" in a stronger and more general sense than the meaning we have given to these words in our axiom. Moreover, assuming $I^{NP}(1) = 0$; this property together with zero normalization implies both the constant sensitivity to null players axiom and the null players equivalence principle. Thus this property can replace both axioms in the characterization of I^{NP}.

It is worth noting that if we choose instead, for each n, $K_n = \frac{1}{n}$ as the value of the constants, we obtain the usual, in other contexts, Gini index. It does not, however, obey any clear equivalence principle. We get in this case:

$$I(1, 0_n) = \left(\frac{n^2}{n^2 - 1}\right) I(1, 0_{n-1}).$$

Note also that when the number of players is large, the Gini index is very close to I^{DP}.

5 INEQUALITY INDICES FOR BANZHAF POWER PROFILES

Let us consider now the case of power profiles generated by the Banzhaf semivalue. In this case the framework for each number n of players is:

$$Co(SG_n) \xrightarrow{Bz_n} Co(Bz_n(SG_n)) \xrightarrow{J_n} \mathbb{R}.$$

So, now the generated set of profiles is $Co(Bz_n(SG_n))$, as mentioned before, a symmetric, compact and convex subset of $\mathbb{R}^n_+$ that strictly contains the $(n-1)$-simplex Δ_n. Thus, now the set of feasible profiles is not a simplex, nor $(n-1)$-dimensional either. For instance, even in the case of only three players, there are Banzhaf profiles whose components add up to more than one, and less in other cases. So, an inequality index in this context is a function that associates a number with each power profile in this set. Again for any such an index J_n we have a composite index $J_n \circ Bz_n$ that ranks games in $Co(SG_n)$.

In principle similar arguments to those used in Section 3 would motivate the assumptions of anonymity, constant sensitivity to null players and zero normalization for an inequality index J_n, now applied to profiles in the new domain $Co(Bz_n(SG_n))$. But now the domain is wider, and the wider the domain the stronger any requirement on the objects of the domain. This is specially so in the case of expected inequality on co-ranked profiles. As we will see later on, this assumption in this domain, though meaningful, restricts too much the set of indices. So, instead, we will require only a restricted form of this condition. Recall the sum of the components of a Banzhaf profile can be interpreted as a democratic participation index of the decision-making process. We require convex linearity only on pairs of co-ranked power profiles with identical democratic participation index. We have thus the following axiom:

Restricted Expected Inequality on Co-ranked profiles (RCLC): For any pair of co-ranked power profiles $\beta, \beta' \in Co(Bz_n(SG_n))$ such that $\sum_{i \in N} \beta_i = \sum_{i \in N} \beta'_i$; and all $\lambda \in [0,1]: J_n(\lambda\beta + (1-\lambda)\beta') = \lambda J_n(\beta) + (1-\lambda)J_n(\beta')$.

Now we need some axiom relating the inequality index of power profiles with different "democracy indices", for none of the former axioms does. In the literature on inequality, a distinction is often made between relative and absolute indices, depending on which is considered relevant, the ratios or the differences between the components of any profile. Note that this distinction was meaningless for the Shapley-Shubik power profiles whose components always add up to 1. These two principles can be expressed as follows:

Relative Index (RI): For all pairs of power profiles $\beta, \lambda\beta \in Co(Bz_n(SG_n))$; $(\lambda \in \mathbb{R})$: $J_n(\lambda\beta) = J_n(\beta)$.

Absolute Index (AI): For all pairs of power profiles $\beta, \beta + \lambda e^N \in Co(Bz_n(SG_n))$; $(\lambda \in \mathbb{R})$: $J_n(\beta + \lambda e^N) = J_n(\beta)$.

Each of these principles, together with the former axioms, will allow us to characterize two inequality indices.

Theorem 5 There is a unique (up to a positive proportionality constant K_n) absolute (respectively, relative) inequality index $J_n : Co(Bz_n(SG_n)) \to \mathbb{R}$ satisfying restricted expected inequality on co-ranked profiles, constant sensitivity to null players and zero normalization. They are respectively given by:

$$Ja_n(\beta_1, \ldots, \beta_n) = K_n \sum_{k=1}^{n} (n - 2k + 1)\hat{\beta}_k, \tag{8}$$

$$Jr_n(\beta_1, \ldots, \beta_n) = K_n \sum_{k=1}^{n} (n - 2k + 1)\frac{\hat{\beta}_k}{\sum_{i \in N} \beta_i}. \tag{9}$$

Proof: First, it is straightforward to check that Ja_n is an absolute index (AI) and Jr_n a relative index (RI). It is also immediate to check that both satisfy REIC, CSNP and ZN. Now, let J_n be an absolute index satisfying the other three conditions in $Co(Bz_n(SG_n))$. Note that Δ_n is contained in $Co(Bz_n(SG_n))$, and REIC implies EIC on Δ_n. Then, by Theorem 3, such an index, satisfying EIC, CSNP and ZN in Δ_n, must be given by (8) on Δ_n. It is only left to be shown that this formula is valid for any profile in the domain. So, let β be a profile in $Co(Bz_n(SG_n))$ such that do not exist $\beta' \in \Delta_n$ and $\lambda \in \mathbb{R}$ such that $\beta = \beta' + \lambda e^N$ (otherwise, by AI, it is immediate). Then, denoting $\eta(\beta) := \sum_{k=1} \beta_k$, it must be $\eta(\beta) > 1$. As $Co(Bz_n(SG_n))$ is symmetric and convex, it contains $\eta(\beta)e^N$. Then, for $\mu \in (0, 1)$ sufficiently close to 0, it will be $\mu\beta + (1 - \mu)\eta(\beta)e^N + (1 - \eta(\beta))e^N \in \Delta_n$. Then, applying first ZN and AI, then REIC, and again AI, we have:

$$\begin{aligned}
\mu J_n(\beta) &= \mu J_n(\beta) + (1 - \mu)J_n(\eta(\beta)e^N) = J_n(\mu\beta + (1 - \mu)\eta(\beta)e^N) \\
&= J_n(\mu\beta + (1 - \mu)\eta(\beta)1^N + (1 - \eta(\beta))e^N + (\eta(\beta) - 1)e^N) \\
&= J_n(\mu\beta + (1 - \mu)\eta(\beta)e^N + (1 - \eta(\beta))e^N) \\
&= J_n(\mu\beta + (1 - \mu\eta(\beta))e^N).
\end{aligned}$$

The last profile belongs to Δ_n, so that (8) can be applied. Thus, we have $\mu J_n(\beta) = \mu K_n \sum_{k=1} (n - 2k + 1)\hat{\beta}_k$, that is, $J_n(\beta) = Ja_n(\beta)$. Finally, let J_n be a relative index satisfying the other three conditions in $Co(Bz_n(SG_n))$. Now the proof is immediate: as before, by Theorem 2, the index must be given by (8) on Δ_n (note in Δ_n (8) and (9) coincide). For any profile β in $Co(Bz_n(SG_n))$, it is $\beta/\phi(\beta) \in \Delta_n$. Then, by RI, we have $J_n(\beta) = J_n(\phi(\beta)(\beta/\phi(\beta))) = J_n(\beta/\phi(\beta)) = Jr_n(\beta)$.

∎

It can be shown that (unrestricted) expected inequality on co-ranked profiles, constant sensitivity to null players and zero normalization (extended to all flat profiles) characterize Ja_n on $Co(Bz_n(SG_n))$. Therefore, requiring expected inequality on co-ranked profiles on $Co(Bz_n(SG_n))$ implicitly implies the choice of an absolute index. It is in this sense that we have said that expected inequality on co-ranked profiles is too strong an assumption in this wider domain.

Now we turn our attention to a general index $J : \bigcup_n Co(Bz_n(SG_n)) \to \mathbb{R}$ to deal with different numbers of players. The situation is similar to that in the previous section: only the zero normalization connects the value of the index for different number of players, establishing, together with the relative (resp., absolute) index axiom, a "common zero" for all flat profiles. So, assuming either a relative index or an absolute index and the other three axioms, there is still a constant K_n undetermined for each number of players. Again, we can use any of the two equivalence principles used with the same purpose in the previous section. Thus, depending on the relative or absolute character of the index and the equivalence principle used, four different indices arise.

Theorem 6 There is a unique (up to a positive proportionality constant K) absolute inequality index: $\bigcup_n Co(Bz_n(SG_n)) \to \mathbb{R}$ satisfying restricted expected inequality on co-ranked profiles, constant sensitivity to null players, and zero normalization for any n, and satisfying the dictator player equivalence principle (respectively the null players equivalence principle). They are respectively given by:

$$Ja^{DP}(\beta_1; \ldots; \beta_n) = K(\tfrac{1}{n-1}) \sum_{k=1}^{n} (n - 2k + 1)\hat{\beta}_k, \quad (10)$$

$$Ja^{NP}(\beta_1; \ldots; \beta_n) = K \sum_{k=1}^{n} (n - 2k + 1)\hat{\beta}_k. \quad (11)$$

Theorem 7 There is a unique (up to a positive proportionality constant K) relative inequality index: $\bigcup_n Co(Bz_n(SG_n)) \to \mathbb{R}$ satisfying restricted expected inequality on co-ranked profiles, constant sensitivity to null players, and zero normalization for any n, and satisfying the dictator player equivalence principle (respectively the null players equivalence principle). They are respectively given by:

$$Jr^{DP}(\beta_1; \ldots; \beta_n) = K(\tfrac{1}{n-1}) \sum_{k=1}^{n} (n - 2k + 1)\frac{\hat{\beta}_k}{\sum_{i \in N} \beta_i}, \quad (12)$$

$$Jr^{NP}(\beta_1;\ldots;\beta_n) \quad = \quad K \sum_{k=1} (n - 2k + 1)\frac{\beta_{(k)}}{\sum_{i \in N} \beta_i}. \qquad (13)$$

It is worth remarking that despite the apparent perfect symmetry between the characterizations of both pairs of absolute and relative indices, there are some important differences concerning the meaning of the equivalence principles, or more precisely, their consequences, in the presence of the remaining assumptions.

Let us first consider the dictator player equivalence principle, which states that the degree of inequality is identical in all dictatorships, whatever the number of players. As noted in the previous section, in the case of Shapley-Shubik power profiles, this principle entails a "common maximum" degree of inequality, that is reached when there is a dictator, whatever the number of players. Similarly, in the case of Banzhaf power profiles the relative inequality index is maximal when there is a dictator. Therefore, the dictator player equivalence principle also entails a "common maximum" of inequality for any number of players in the case of the relative index. But this is not true for the absolute inequality index: there exist simple superadditive games[2] whose Banzhaf profiles lead to a larger absolute index of inequality than the index of a dictatorship with the same number of players. This fact, intimately related to the absolute character of the index, makes the interpretation of this equivalence principle less intuitive.

Now, let us turn our attention to the null player equivalence principle. As noted in the previous section, in the context of Shapley-Shubik profiles, this is a particular case of a more general principle stating that profiles with different number of players in which the power is equally shared by a group of them, are considered as equivalent from the inequality point of view if the number of null players is the same. It can be seen that this general principle continues to be valid for the relative index, but no more for the absolute index. This general principle does not even hold any more for a fixed number of players, as illustrated in the following example[3]:

$$Ja^{NP}\left(\frac{1}{4};\frac{1}{4};\frac{1}{4};0\right) = \frac{3}{4} \neq \frac{3}{2} = Ja^{NP}\left(\frac{1}{2};\frac{1}{2};\frac{1}{2};0\right).$$

Finally, let us consider the constant sensitivity to null players. It is easy to check that

[2] For instance, let u be the compound game (see Owen (1982) for a definition) in which the first stage games are the simple majority games with $7; 3; 5$ and m_4 players, respectively, and the second stage game is the 4-person simple game in which the only minimal coalitions are $\{1; 2\}$ and $\{1; 3\}$. It can be checked that $Ja(Bz(u)) > Ja(1; 0_{14+m_4})$ whenever $m_4 > 5$.

[3] The first Banzhaf profile corresponds to a game where any coalition containing the first three players is winning, while the second corresponds to a game where any coalition containing at least two of the first three players is winning.

Ja^{NP} and Jr^{NP} satisfy the following equations:

$$Ja^{NP}(\beta_1, \ldots, \beta_n, 0) = Ja^{NP}(\beta_1, \ldots, \beta_n) + K \sum_{k=1} \beta_k, \tag{14}$$

$$Jr^{NP}(\beta_1, \ldots, \beta_n, 0) = Jr^{NP}(\beta_1, \ldots, \beta_n) + K, \tag{15}$$

where K is the constant that appears in (11) or (13). Again, in the case of the relative index, this property could be properly called "constant sensitivity to null players" in the stronger and more general sense given in the previous section. This property can also replace the constant sensitivity to null players axiom and the null players equivalence principle in the characterization of Jr^{NP}. But let us examine the situation underlying formulae (7), (14) and (15). In fact, this corresponds to the addition of a null player to a game. Indeed, if we consider two games (N, v) and (N', v'), with $N' = N \cup \{n+1\}$ and $v'(S) = v(S \cap N)$ for any coalition $S \subseteq N'$, it follows from formulae (1) and (2) that $Sh_{n+1}(v') = (Sh_n(v), 0)$ and $Bz_{n+1}(v') = (Bz_n(v), 0)$. That is, the effect in the power profile is just adding one zero for the new component, the rest continuing to be the same. Thus these formulae yield:

$$\begin{aligned} I^{NP}(Sh_{n+1}(v')) &= I^{NP}(Sh_n(v)) + K \ , \\ Ja^{NP}(Bz_{n+1}(v')) &= Ja^{NP}(Bz_n(v)) + K \frac{\eta(v)}{2^{n-1}}, \\ Jr^{NP}(Bz_{n+1}(v')) &= Jr^{NP}(Bz_n(v)) + K. \end{aligned}$$

These equations reflect through our inequality indices some differences between the Shapley value and the Banzhaf semivalue used as power indices and between the absolute and the relative inequality indices. In fact, the impact of adding a null player on Ja^{NP} is not constant, as it is on I^{NP} or Jr^{NP}. It depends on the game the null player joins. To illustrate it, let us consider two symmetric decision-making processes: a unanimity rule and a simple majority rule. Each player's Shapley value is identical in both games (by the constant total power axiom), while each player's Banzhaf semivalue is larger in the simple majority game than in the unanimity game. The inequality indices are, however, identical in all cases and equal to zero. The introduction of a null player in both games changes in both cases the inequality index from zero to K if the Shapley profiles are considered. The result is the same for the relative inequality index with the Banzhaf profiles while the absolute inequality index varies from zero to $K \frac{\eta(v)}{2^{n-1}}$. Therefore the impact of adding a null player is bigger with regard to the absolute inequality in the simple majority rule than in the unanimity rule. This reflects that the difference in terms of power between the null player and the others is larger in the simple majority rule than in the unanimity rule. This seems consistent with Dubey and Shapley's interpretation of $\eta(v)$ as a "democratic participation index".

Finally, observe that, if for any Banzhaf profile β, β_a and β_r denote, respectively, the additive and the multiplicative normalization of β; then $J_a(\beta) = I(\beta_a)$ and $J_r(\beta) = I(\beta_r)$, if I is defined by formula (4) on the hyperplane $\sum_{i \in N} \beta_i = 1$ (and this for each of the variants of these indices).

6 ILLUSTRATION: THE U.N. SECURITY COUNCIL

As an illustration of the computation and working of the inequality indices introduced in Sections 4 and 5, we apply them to compare the two different decision processes, before and after 1965, of the UN Security Council.

Since the creation of the Security Council, in 1945, up to 1965, decisions on issues of substance required the approval of its 5 permanent members and at least 2 of its 6 non-permanent members. This procedure was often criticized because of the excessive power given to the five permanent members. In 1965, in order to reduce the power of the permanent members, the number of non-permanent members was augmented to 10, and decisions required, in addition to that of the 5 permanent members, the positive vote of 4 of the 10 non-permanent members. The effectiveness of this reform has been critically analyzed with different approaches (see, e.g., Riker and Ordeshook (1973) and Winter (1996)). We just apply our inequality indices to both Shapley-Shubik and Banzhaf power profiles of the following 11 and 15-person games that formally describe both decision processes.

Before 1965: let $N = P \cup T$ be the set of players, where P denotes the permanent members ($p = 5$), and T denotes the non-permanent members ($t = 6$). Then

$$v(S) = \begin{cases} 1 & \text{if } P \subset S \text{ and } s \geq 7, \\ 0 & \text{otherwise.} \end{cases}$$

After 1965: let $N' = P \cup T'$ be the set of players, where P denotes the permanent members ($p = 5$), and T' denotes the non-permanent members ($t' = 10$). Then

$$v'(S) = \begin{cases} 1 & \text{if } P \subset S \text{ and } s \geq 9, \\ 0 & \text{otherwise.} \end{cases}$$

The power profiles are respectively given by

	Before 1965 $(N; v)$	After 1965 $(N'; v')$
Sh(permanent)	0.1974	0.1963
Sh(non-permanent)	0.0022	0.0019
Bz(permanent)	0.0557	0.0517
Bz(non-permanent)	0.0049	0.0051

Applying formulae (5, 6, 10, 11, 12, and 13), we respectively get:

	Before 1965 (N,v)	After 1965 $(N^0; v^0)$
$I^{DP}(Sh)$	K 0:5857	K 0:6943
$I^{NP}(Sh)$	K 5:8571	K 9:7203
$Ja^{DP}(Bz)$	K 0:1523	K 0:1665
$Ja^{NP}(Bz)$	K 1:5234	K 2:3315
$Jr^{DP}(Bz)$	K 0:4952	K 0:5371
$Jr^{NP}(Bz)$	K 4:9524	K 7:5197

It is remarkable the coincidence in the assessment of the comparative degree of inequality: all inequality indices, either based on the dictator player equivalence principle or based on the null players equivalence principle, either applied to Shapley-Shubik or to Banzhaf pro¯les, either absolute or relative in this case, rank both decision-making processes in the same way: after 1965 the inequality has increased. This seems contradictory with the supposed aim of the reform. But as we have mentioned before, doubts about its e®ectiveness have been already raised. Winter (1996) points out two opposite e®ects of the reform: "On the one hand, it becomes harder for veto players to form a winning coalition because that requires the supporting votes of more non veto members. On the other hand, the power of non veto members may be reduced since each such member now has more substitutes than before." So, the permanent members' power decreases, but in the whole our indices evaluate that from the inequality point of view the situation has been worse since 1965.

7 CONCLUDING REMARKS

As we say in the introduction, this paper is meant as a ¯rst step to provide an axiomatic support to some inequality indices to deal with comparisons of voting procedures according to the degree of inequality in the distribution of power. To do so we have tried to put forward conditions that make sense in terms of the involved concept of power in voting systems. We want to stress some positive points and some di±culties, as well as pointing out some lines of further research along the two approaches discussed in the introduction.

We want to emphasize the meaningfulness of the underlying domain of games that we propose, that is, the convex hull of simple superadditive games. This domain, interpreted as the set of probabilistic mixtures of simple superadditive games is a natural extension of the usual domain of simple superadditive games as formal descriptions of voting procedures. In connection with our endeavor, it is worth stressing two points. First, this underlying choice gives a clear support in this context to our assumption of (restricted or not) "expected inequality on co-ranked pro¯les". Second, signi¯cantly, this one seems to be the only domain where Einy and Peleg's work could be meaningfully restated in the context of distribution of power. Indeed, on the one hand, the class of TU-games that they consider goes too far beyond the models of voting rules. On the other hand, in the domain of simple superadditive games that is usually used to model voting rules, the axioms that they propose do not make sense (because the addition of two simple games is not a simple game). In the domain that we propose the worth of a coalition can be interpreted as the probability of being winning, and the axioms they propose make sense if convex combination of games is taken instead of addition of games.

We have extended Dubey and Shapley's axiomatizations of the two best-known power indices to this domain. Then, taking the corresponding power pro¯les as primitives, we have axiomatized some measures of inequality in the distribution of power. Consequently, the choice of one of our inequality indices requires the previous choice of a power index. This choice may depend on the context (Laruelle, 1999), but to evaluate the a priori capacity to in°uence the outcome of a vote in a given voting rule, the Banzhaf semivalue seems more suitable than the Shapley-Shubik index (and any other existing power indices). However, the results concerning the Shapley-Shubik pro¯les seem more solid because the absence of the absolute/relative dichotomy raises no doubts. Instead, when dealing with Banzhaf pro¯les this issue may raise some doubts. Indeed, as discussed in the last few paragraphs of Section 5, both "equivalence principles", as well as the "constant sensitivity to null players" have a more clear meaning for a relative inequality index than for an absolute one. In this sense, the indices I and J_r seem to be better founded. Concerning the practical applications of these tools, in the example considered in Section 6 the message transmitted by all indices go in the same direction. However it remains to be checked whether this is often the case or not.

With respect to the second approach, using the simple games (or lotteries over them) as primitives, we claim that the mechanical application of Einy and Peleg's results do not make sense. This approach, maybe more ambitious, is still an interesting line for further research to be carried out in this speci¯c context. Here we would like to stress again the speci¯city of simple superadditive games when they are used to model decision-making procedures. Indeed, if simple superadditive games are a subdomain of TU-games, compelling intuitions for TU-games do not necessarily remain intuitive when they are interpreted as decision-making processes. For instance, if Einy and Peleg's work appears well-founded for the general class of TU-games, they implicitly take for granted e±ciency, which seems indeed quite natural in many contexts. But in the context of decision-making processes, the e±ciency implicit in their independence axiom may lead to some counterintuitive results. Restated in our domain, this axiom would say that for any games $u, v, w \in Co(SG_n)$ such that u and v are T-symmetric[4] for some coalition T, and any λ $(0 \leq \lambda \leq 1)$, it must be: $\lambda u + (1 - \lambda)w \sim \lambda v + (1 - \lambda)w$. But the following example shows how counterintuitive some of the consequences of this axiom can be in the context of decision making processes. Let $N = \{1, 2, ..., 6\}$; $T = \{1, 2, 3\}$, u and w the unanimity games $u = u^T$, $w = u^{N \setminus T}$ and v the simple superadditive game whose winning coalitions are those containing at least two players of coalition T. As u and v are T-symmetric games, any binary relation on $Co(SG_n)$ satisfying the adapted IND must yield: $1/2u^T + 1/2u^{N \setminus T} \sim 1/2v + 1/2u^{N \setminus T}$. And while intuition (as just anonymity) compellingly suggests that in the left hand side lottery equality is perfect, this is not the case in the right hand side lottery, where the power of any player in T seems prima facie different from that of any player in $N \setminus T$. In fact, this example yields some interesting conclusions. First, it shows that the validity of the above sketched translation of Einy and Peleg's results to our domain is quite questionable. Therefore, their program, according to which the axiomatic foundation of inequality should take as primitives the games instead of the profiles associated to them by any particular solution, is still to be re-thought from the beginning in the context of collective decision processes. Second, the direct intuition provided by this example can be held critically against the suitability of the Shapley value as a measure of power in collective decision processes: this measure associates identical power profiles with both lotteries, and, a fortiori, any inequality measure that explicitly or implicitly embodies this index would identify them from the point of view of inequality, against the direct assessment provided by intuition. It is remarkable how in this example both concepts, power and inequality, or, better, the clear and direct intuition of them at a pre-formal level, conflicts with the use of the Shapley value as a measure of power.

[4] A T-symmetric game is a game in which all players outside coalition T are null players, while all players inside T are substitutes.

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