A TOBIT MODEL WITH GARCH ERRORS[1]

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ABSTRACT

In the context of time series regression, we extend the standard Tobit model to allow for the possibility of conditional heteroskedastic error processes of the GARCH type. We discuss the likelihood function of the Tobit model in the presence of conditionally heteroskedastic errors. Expressing the exact likelihood function turns out to be infeasible, and we propose an approximation by treating the model as being conditionally Gaussian. The performance of the estimator is investigated by means of Monte Carlo simulations. We find that, when the error terms follow a GARCH process, the proposed estimator considerably outperforms the standard Tobit quasi maximum likelihood estimator. The efficiency loss due to the approximation of the likelihood is finally evaluated.

Key Words: censored regressions; conditional heteroskedasticity; Monte Carlo simulations.

JEL classification numbers: C13, C15, C22, C24.

1 Introduction

In empirical applications we often encounter examples where the dependent variables in regression models can be observed only in a limited range. As pointed out by Maddala (1983, p.1), it is not always necessary to introduce the complications implied by this type of model. However, when the variables are limited in their range because of some underlying stochastic choice mechanism, the limited-dependent variable (LDV) models provide the suitable theoretical framework.

The first application of an LDV model to economic problems was proposed in a pioneering work by Tobin (1958). Analyzing demand for durable goods, he found that most households report zero expenditure on automobiles or other durable goods during the year. The linearity assumption underlying the regression model was clearly inappropriate, and some suitable form of discontinuity had to be introduced. This led to a *censored regression model* and, given the strict connection with the literature on Probit models, Goldberger (1964) introduced the term *Tobit* to synthesize in one word the concept "Tobin's Probit".

In subsequent years the simple censored regression, which is also referred to as *standard* Tobit model, was generalized and more complicated versions of it are now routinely used in various fields of economics and social sciences (see Amemiya, 1984).

The use of LDV models is certainly more frequent in the microeconometric analysis of survey data. Nevertheless, recent examples of LDV applications can be found also in time series models.

Bank of Italy (1988) reports the results related to the estimation of the demand of discount window borrowing (see also Calzolari and Fiorentini, 1993). Edin and Vredin (1993) and Garcia (1994) apply standard LDV models to target zone exchange rates.

Particularly relevant is the limited-dependent variable rational expectations (LDV-RE) model of Pesaran and Samiei (1992a) used to explain the exchange rate behaviour and the devaluation risk in a target zone (see also Baxter 1987, and Bertola and Svensson 1990). Other applications of LDV-RE models can be found in Lee (1994), Pesaran and Ruge-Murcia (1993, 1995) and Pesaran and Samiei (1992b, 1995). Beside their use in the empirical analysis of the exchange rate target zone mechanism, these models are apt to describe the behavior of the dependent variable in government-regulated markets. Examples are commodity price support schemes and targeted interest rates. In addition, when agent expectations are involved, the LDV approach is important even when the endogenous variable remains within the bounds.

What the above investigations on time series LDV models have in common is the nature of the variables being analyzed. They concentrate on the behaviour of financial and monetary series, and it is well known that the assumption of normality is usually not appropriate for this kind of series as they rather seem to follow some thick-tailed distribution. As a consequence, financial and monetary variables are usually better modelled assuming a conditional heteroskedastic error process of the ARCH type (see Engle 1982 or, for a survey, Bollerslev, Engle, and Nelson 1994).

Contrary to the linear regression model, ignoring conditional heteroskedasticity in a Tobit model rises several problems. For instance, Maddala and Nelson (1975) show that, if we ignore heteroskedasticity, the resulting estimates are not consistent. Also Hurd (1979), and Arabmazar and Schmidt (1981) investigate the properties of the Tobit maximum likelihood estimator when the error process is heteroskedastic. Under non-normality of the disturbances, the Gaussian quasi-maximum likelihood (QML) Tobit estimator yields inconsistent estimates of the parameters (e.g. Arabmazar and Schmidt, 1982).

This motivates our proposal of a Tobit model with conditionally heteroskedastic errors which is developed in this paper. The main difficulty is discussed in Pesaran and Ruge-Murcia (1995):

> " ... notice that the limited-dependent nature of the endogenous variable y_t makes the exact calculation of the residuals for the censored observation infeasible. The difficulty arises because for the case of observations at the bound, the exact values of the residuals are not observed by the econometrician. Thus, it does not seem viable ... "

In what follows we propose a Tobit model with GARCH errors which is described in details in Section 2 where we also discuss the *feasible* approximated maximum likelihood estimator. Section 3 displays the results of some Monte Carlo simulations which exemplify the improved behavior of the Tobit-GARCH estimator over the standard Tobit-ML estimator. We also notice that a Tobit-GARCH-ML estimator, based on knowledge of past disturbances, can be defined but it turns out to be infeasible with real data. Its feasibility is confined to simulated data, and therefore, using simulation, this optimal estimator can be used as a benchmark to evaluate the efficiency loss in our feasible estimator. Section 4 draws the conclusion.

2 The Model

The censored regression (or Tobit) model is defined as

$$y_t^\star = x_t'b + u_t \qquad \begin{cases} y_t = y_t^\star & \text{if RHS} > 0 \\ y_t = 0 & \text{if RHS} \leq 0 \end{cases} \tag{1}$$

where $y_t^\star$ is an unobservable random variable, x_t is the vector of exogenous explanatory variables at time t, b is the vector of unknown coefficients, y_t is the observed *censored* value of the dependent variable. A typical assumption is that of independent identically distributed normal error terms u_t. However, when financial or monetary variables are involved, it is more realistic to assume that u_t follows a conditional heteroskedastic process of the ARCH class.

The definition of a Tobit model with conditional heteroskedastic disturbances raises several problems. Primarily because the conditional distribution of the error terms may depend on the entire past history of the process that, in a censored regression, is only partially observed by the econometrician. In order to give a clear picture of the issues related to the definition of such models we need to carefully discuss various possibilities concerning processes and estimators. In this respect, let $\mathcal{I}_t \equiv \{y_t^*, x_t, y_{t-1}^*, x_{t-1}, \ldots\}$ denote the information set that embodies the past realization of the process. If censoring occurs, the information set available to the econometrician will be smaller than $\mathcal{I}_t$. For this reason, we also need to define $\Psi_t \equiv \{y_t, x_t, y_{t-1}, x_{t-1}, \ldots\}$ as the econometrician information set.

We first consider the standard case of normality and constant variance (i.e. $u_t \sim N(0, \sigma^2)$). Since the distribution of the disturbances u_t conditional on the available information Ψ_{t-1} is censored normal (e.g. Amemiya 1984), the log-likelihood function is given by

$$\log L(b, \sigma) = \sum_0 \log(1 - \Phi_t) - \frac{T_1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_1 (y_t - x_t' b)^2 , \qquad (2)$$

where

$\Phi_t \equiv$ cumulated distribution function of a standard normal evaluated at $x_t' b/\sigma$;

$\sum_0 \equiv$ summation referring to the zero observations;

$\sum_1 \equiv$ summation referring to non-zero observations;

$T_1 \equiv$ number of nonzero observations.

Let us now move to conditional heteroskedastic processes but, for the moment, assume that the variance is some known function of exogenous variables as in

$$u_t \mid \mathcal{I}_{t-1} \sim N(0, \sigma_t^2) \, ; \qquad\qquad \sigma_t^2 = \delta_0 + \delta_1 x_{j,t-1}^2 \tag{3}$$

for some $x_j \in \mathcal{I}$. This model specification implies no additional complications with respect to the standard case since, given the parameters, the conditional variance is measurable also in Ψ_{t-1}. Then, the log-likelihood function becomes

$$\log L(b, \delta) = \sum_0 \log(1 - \Phi_t) - \frac{1}{2} \sum_1 \log \sigma_t^2 - \frac{1}{2} \sum_1 \frac{(y_t - x_t' b)^2}{\sigma_t^2} \, , \tag{4}$$

where $\delta' = (\delta_0, \delta_1)$ and Φ_t is now evaluated at $x_t' b / \sigma_t$. This standard approach to heteroskedasticity requires a specification of the causes of the changing variance which are often not straightforward to determine.

A preferable model is the (strong) Garch(p,q) process, i.e.

$$u_t \mid \mathcal{I}_{t-1} \sim N(0, \sigma_t^2) \, , \qquad\qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \, . \tag{5}$$

Notice that, because of censoring, the past values of the error term u_t are not directly observable and, as a consequence, the distribution of u_t conditional on the econometrician information set Ψ_{t-1} is not (censored) Gaussian. We have that $\Psi_{t-1} \subset \mathcal{I}_{t-1}$, and the conditional distribution of the disturbances, $u_t \mid \Psi_{t-1}$, cannot be determined readily. A simple example is given by the extreme situation in which the conditioning set is the empty set, the conditional distribution is then equal to the marginal. It is well known, from Arch theory, that this marginal distribution function is unknown, thick tailed, with zero mean and variance that is easily obtained from the parameters of the Arch process.

In practice Ψ_{t-1} will not be empty and will convey a lot of information, especially when the degree of censoring is not severe, it is thus worth to incorporate this information into the likelihood function. In order to get a feasible approximation of the log-likelihood, we may proceed, as in Harvey, Ruiz and Sentana (1992), on the basis that the model can be treated as though it were conditionally (censored) normal and we will refer to the resulting log-likelihood function as the Tobit-ARCH approximate log-likelihood, given by

$$\log L^{\star}(b,\alpha,\beta)=\sum_{0}\log(1-\Phi_t)-\frac{1}{2}\sum_{1}\log(\tilde{\sigma}_t^2)-\frac{1}{2}\sum_{1}\frac{(y_t-x_t'b)^2}{\tilde{\sigma}_t^2} \tag{6}$$

where $\alpha'=(\alpha_0,\dots\alpha_q)$, $\beta'=(\beta_1,\dots\beta_p)$ and Φ_t is now evaluated at $x_t'b/\tilde{\sigma}_t$. In the above expression, $\tilde{\sigma}_t^2$ is computed from

$$\tilde{\sigma}_t^2=\alpha_0+\sum_{i=1}^{q}\alpha_i\tilde{u}_{t-i}^2+\sum_{j=1}^{p}\beta_j\tilde{\sigma}_{t-j}^2\,. \tag{7}$$

It only remains to determine $\tilde{u}_{t-i}^2$. For this purpose, we have to distinguish between censored and observed y_{t-i}. When the dependent variable is observed then u_{t-i}^2 is in Ψ_{t-i} and we take $\tilde{u}_{t-i}^2=u_{t-i}^2=(y_{t-i}-x_{t-i}'b)^2$. When y_{t-i} is censored we define $\tilde{u}_{t-i}^2$ as an approximation to the expected value of u_{t-i}^2 calculated by proceeding as if u_{t-i} were conditionally Gaussian zero mean and variance $\tilde{\sigma}_{t-i}^2$, in which case it can be easily shown (see Johnson and Kotz, 1970) that

$$\tilde{u}_{t-i}^2=\widetilde{E}_{t-i}(u_{t-i}^2\mid y_{t-i}^{\star}\le 0)=\tilde{\sigma}_{t-i}^2-(x_{t-i}'b)\widetilde{E}_{t-i}(u_{t-i}\mid y_{t-i}^{\star}\le 0)\,, \tag{8}$$

where

$$\widetilde{E}_{t-i}(u_{t-i}\mid y_{t-i}^{\star}\le 0)=-\frac{\tilde{\sigma}_{t-i}\phi_{t-i}}{1-\Phi_{t-i}}\,. \tag{9}$$

Note that by doing so we introduce another source of approximation. The next section is devoted to assess, by means of simulation, whether all these approximations are of great importance in practice.

Perhaps, it is also noteworthy to point out that if we modified slightly the assumptions about the data generation process, then $\log L^{\star}(b,\alpha,\beta)$ could be seen as the exact log-likelihood function. In this respect consider the following conditional heteroskedastic process for the disturbances u_t, that is

$$u_t\mid\Psi_{t-1}\sim N(0,\tilde{\sigma}_t^2);\qquad \tilde{\sigma}_t^2=\alpha_0+\sum_{i=1}^{q}\alpha_i E_{t-i}(u_{t-i}^2\mid\Psi_{t-i})+\sum_{j=1}^{p}\beta_j\tilde{\sigma}_{t-j}^2\,. \tag{10}$$

Harvey, Ruiz and Sentana (1992) discuss and derive the statistical properties of processes like the one in (10). Unless there is some compelling reason to assume such a data generation process, we think it is preferable to stay with the (strong) GARCH(p,q) process of equation (5) and interpret $\log L^\star(b, \alpha, \beta)$ as an approximation to the unknown log-likelihood function.

The approximated maximum likelihood estimator of the Tobit-GARCH model is given by the values of $\hat{b}$, $\hat{\alpha}$, and $\hat{\beta}$ that maximize $\log L^\star(b, \alpha, \beta)$. The parameter estimates can then be computed with a computationally efficient mixed-gradient algorithm (analogous to the one described in Fiorentini, Calzolari and Panattoni, 1996), which in early iterations of the quasi-Newton maximization procedure, makes use of the matrix of outer products of the first derivatives of the log-likelihood, then switches to the Hessian matrix and uses it until convergence to a maximum is reached.

Unlike standard Tobit models (without GARCH errors), or standard GARCH models (without limited dependent variables), the analytical derivatives of $\log L^\star(b, \alpha, \beta)$ are extremely burdensome to compute in our case. We have therefore employed numerical derivatives to approximate their analytical counterparts.

For hypothesis testing purposes, the inverse of the Hessian matrix (with minus sign) of the approximated log-likelihood function will be used as an estimator of the covariance matrix of the parameter estimates. Let θ denote the vector of all parameters of the model, $\theta = (b', \alpha', \beta')'$; our approximation to the finite sample covariance matrix of θ has been computed as

$$\text{Cov}\ \left(\hat{\theta}\right) \ = \ \left(- \ \frac{\partial^2 log L^\star(\theta)}{\partial\theta \quad \partial\theta'} \right)^{-1}_{\theta=\hat{\theta}} . \tag{11}$$

Since $\hat{\theta}$ is obtained by maximizing an approximate likelihood function it would be natural to follow White (1982) and compute the QML covariance estimator. Calzolari and Fiorentini (1996) adopt this estimator and results are not significantly different from those obtained with the expression in (11).

3 Simulation Results

In order to investigate the finite sample performance and potential applicability of the Tobit-GARCH estimator discussed above, some Monte Carlo simulation experiments were conducted.

To facilitate the presentation, all the simulated models were nested within the following specification

$$y_t^\star = b_0 + b_1 x_{1,t} + b_2 x_{2,t} + b_3 x_{3,t} + u_t \tag{12}$$

$$u_t \mid \mathcal{I}_{t-1} \;\sim\; N(0, \sigma_t^2)\,; \qquad \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2\,. \tag{13}$$

The three exogenous variables were generated according to the following zero-mean stable VAR(1) process

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{pmatrix} = \begin{pmatrix} .4 & -.2 & .2 \\ .7 & -.4 & .1 \\ .2 & .7 & .6 \end{pmatrix} \begin{pmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{3,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix}, \tag{14}$$

where

$$\epsilon_t \;\sim\; N(\,0,\,\Sigma) \qquad \Sigma \;=\; \begin{pmatrix} 1 & .7 & .2 \\ .7 & 1 & .5 \\ .2 & .5 & 1 \end{pmatrix}. \tag{15}$$

3.1 Evaluation of the Efficiency Gain

We were interested in evaluating the performance of the Tobit-GARCH estimator with respect to the usual Tobit-QML estimator (which ignores heteroskedasticity) when the error terms in the data generation process follow a conditional heteroskedastic process of the GARCH type (given in (5)). In doing this we concentrated on how this performance depends on the specification of the error process, on the percentage of censored observations in the sample and on the sample length. Therefore, the simulated models differ with respect to the specification adopted for the error process, with respect to the average percentage of censored observations and with respect to the sample length.

For the error terms we have assumed in some experiments an ARCH(1) process with $\alpha_1 = .5$ and unconditional variance equal to 1, and in other experiments a GARCH(1,1) process with $\alpha_1 = .3$, $\beta_1 = .5$ and, again, unconditional variance equal to 1.

The average percentage of censored observations was controlled by setting different values for the intercept coefficient. In particular we set b_0 equal to 20.0, 1.2, 0.38 and -0.38, which imply an average percentage of censored observations equal to 0, 20, 40 and 60 respectively.

Finally, the sample lengths were set equal to 200, 400, and 1000. This makes a total of 24 different model specifications to experiment with. Notice that the values of the slope coefficients ($b_1 = 3$, $b_2 = 2$ and $b_3 = 1$) were kept fixed across the experiments limiting, to some extent, the general validity of the experiments, but permitting a more compact presentation of the results.

For each model specification we performed 2000 replications of the Monte Carlo process in the following way. First, we generated values of the exogenous variables, held fixed over all replications, according to the vector autoregressive scheme of equations (14) and (15). Independently of the explanatory exogenous variables we then generated the conditionally heteroskedastic disturbance terms, u_t, over the sample period, with the given ARCH-GARCH structure. Values of the unobserved endogenous variable, $y_t^\star$, were computed using the vector of true parameters of the regression equation. The observed variable was finally computed as $y_t = \max(0, y_t^\star)$, and was used for estimation of the Tobit-GARCH parameters and of the Tobit quasi-maximum likelihood parameters (the Tobit-QML carried out under the hypothesis of omoskedastic Gaussian error terms).

At the end of the 2000 replications, we computed the following:

- The Monte Carlo means of the parameter estimates, obtained from Tobit-QML and Tobit-GARCH.
- The Monte Carlo means of the estimated variance-covariance matrices of all parameters. These matrices are computed in each Monte Carlo replication from the Hessian of the Gaussian log-likelihood for the Tobit-QML case, and from the Hessian of the approximated log-likelihood (eq. 11) for the Tobit-GARCH. The square root of the mean estimated variance is displayed in parentheses under each parameter.
- The Monte Carlo variances[1] of the estimated parameters. Monte Carlo standard errors are obtained as square roots and are displayed in square brackets under each parameter.
- Mean squared errors over the Monte Carlo replications for the Tobit-QML and for the Tobit-GARCH estimators. Their ratio is displayed in the tables.

If, for a particular sample, the estimation algorithm did not converge the Monte Carlo replication was discarded and ignored. However, non-convergencies were not a serious problem and only occurred for small sample lengths and high censoring percentages with a maximum of roughly 3% for the case of T=200 and 60% censoring.

[1]By Monte Carlo variance (mean squared error) of the parameter estimates we mean the variance (mean squared error) of the sampling distribution of the parameter estimates. Let $\hat{\theta}_i$ denote the parameter estimate at the i-th replication and let n be the number of replications. We define the Monte Carlo variance of $\hat{\theta}$ as $1/n \sum_{i=1}^{n} \hat{\theta}_i^2 - (1/n \sum_{i=1}^{n} \hat{\theta}_i)^2$.

Detailed results of the simulation experiments are displayed in Tables 1 and 3 only for the case of an average 40% of censored observations, being Table 1 related to the ARCH(1) specification of the error terms and Table 3 to the GARCH(1,1).[2] Tables 2 and 4 report, for all the models, the ratios between the MSE of the Gaussian Tobit quasi-maximum likelihood estimates and the MSE of the Tobit-GARCH estimates.

When the true data generating mechanism is Tobit-GARCH, results of Tables 1-4 show that the efficiency gain of our Tobit-GARCH estimator is substantial. The MSE of the standard Gaussian Tobit ML estimates is usually between 30% and 50% larger than the MSE of the Tobit-GARCH estimates (this percentage reaches, on occasions, values that are even much larger).

Tables 1 and 3 tell us that the efficiency of both estimators, measured by the Monte Carlo MSE, increases with the sample length. However, the results of our experiments do not clearly show what is the effect of the sample length and of the censoring on the relative efficiency of the Tobit-GARCH over the standard Tobit-ML. It seems that the MSE ratios of the slope coefficients are invariant to (or possibly decreasing with) the percentage of censoring. This is confirmed by the figures in the first row of Tables 2 and 4 which refer to the case of zero censored observations. On the other hand, when the percentage of censored observations is high, the relative efficiency of the Tobit-GARCH estimator slightly improves as the sample period lengthens.

The results in Tables 1 and 3 also show that the inverse of the Hessian (with minus sign) of the approximated log-likelihood provides a good estimator of the finite samples covariance matrix of parameter estimates.

[2] The results related to the whole set of experiments and the FORTRAN program employed in the Monte Carlo are available on request from the authors.

3.2 Evaluation of the Efficiency Loss

Finally, it would be of interest to know to what extent the use of the approximation to the exact likelihood function affects the performance of the Tobit-GARCH estimator. An idea about the magnitude of the efficiency loss, due to the approximation, can be obtained using a procedure already exploited, in a different set-up, in Harvey, Ruiz and Sentana (1992).

Since in a Monte Carlo simulation the $y_t^\star$ are artificially constructed we can write the likelihood conditioning on $\mathcal{I}_t$ instead of Ψ_t and the fully efficient estimator is given by maximization of

$$\log L(b, \alpha, \beta) = -\frac{1}{2}\sum_{t=1}^{T} \log \sigma_t^2 - \frac{1}{2}\sum_{t=1}^{T} \frac{(y_t^\star - x_t'b)^2}{\sigma_t^2}\,, \tag{16}$$

which is the usual GARCH regression model log-likelihood function. However, we think that this would not provide a suitable basis for comparison because the above estimator is not affected by censoring.

A better candidate is a model like (4) where the data are censored but the conditional variance is measurable with respect to the available information. For this purpose we can define a log-likelihood function based on the distribution of u_t conditional on $\mathcal{A}_t \equiv \{y_t, x_t, y_{t-1}^*, x_{t-1}, y_{t-2}^*, \ldots\}$ in which past disturbances are treated as known but current observations may be censored. This yields

$$\log L(b, \alpha, \beta) = \sum_0 \log(1 - \Phi_t) - \frac{1}{2}\sum_1 \log \sigma_t^2 - \frac{1}{2}\sum_1 \frac{(y_t - x_t'b)^2}{\sigma_t^2}\,, \tag{17}$$

where

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2\,. \tag{18}$$

Maximization of (17) provides consistent parameter estimates and it is only feasible with simulated data. The above model is indeed an artifact but permits a precise assessment of the efficiency loss in the Tobit-Garch estimator that is due to the approximation arising from non measurability of the conditional variance with real data.

The Monte Carlo experiments were re-run, but this time the parameters of the models were estimated with the infeasible Tobit-GARCH estimator defined in (17) (labelled "Tobit infeasible" in the tables) that was then compared with our feasible estimator. Tables 5-8 report the results of these simulations.

Looking at the tables, we observe that the efficiency loss of the Tobit-GARCH estimator with respect to the infeasible one is rather small, unless the censoring percentage and the sample length are both very large.

It is curious to notice that when the sample size and the censoring percentage are small the Tobit-GARCH estimator of the α_1 parameter is more efficient than the corresponding infeasible estimator. One might think that this odd outcome is due to the variance of the Monte Carlo and therefore we repeated those experiments running 15000 replications of the simulation process but the results did not change significantly. Thus, we must conclude that this behavior is due to some small sample effect which is not straightforward to interpret.

A last comment should be made about the inconsistency of the Tobit-GARCH estimator which is evident only for large censoring percentage and, anyhow, is always smaller than in the standard Tobit-QML. Overall the magnitude of the small sample and asymptotic bias of the Tobit-GARCH estimates does not seem to represent a major problem in most practical applications.

Still, by resorting to indirect inference techniques, the Tobit-GARCH estimator proposed in this paper can serve as a basis for constructing a consistent estimator of the parameters of censored regression models with GARCH errors. The approximation to the exact likelihood and the dimension of the parameter vector of the Tobit-GARCH model make it suitable to be used as the "incorrect" criterion in the indirect inference procedure described in Gourieroux, Monfort and Renault (1993).

Indirect inference techniques can solve the asymptotic bias problem but will produce an estimator with larger asymptotic variance than the Tobit-GARCH so that the overall outcome is not clear. This issue is analyzed in Calzolari and Fiorentini (1996). They find that the indirect inference procedure achieves a negligible bias reduction at the cost of a considerable increase in small sample variance of the estimator.

4 Conclusion

Heteroskedasticity of the error terms has long been recognized as a critical condition for the performance of the Tobit ML estimator. When Tobit models are applied to time series regression involving financial and monetary dependent variables, the conditional distribution can be reasonably assumed to follow a process of the GARCH type.

In this paper we propose a feasible approximation to the maximum likelihood estimator of the Tobit-GARCH model. The performance of this estimator are analyzed by means of simulations and the results show that it considerably outperforms the usual Gaussian Tobit-QML estimator.

Still by simulation, we evaluate the efficiency loss with respect to an ideal Gaussian Tobit-GARCH maximum likelihood estimator which is, however, infeasible with real data.

A limitation of this paper is that it only analyses the standard Tobit model, while many time series applications consider, for example, more sophisticated versions of LDV models. However, we believe that this research provides the basis to generalize the Tobit-GARCH model. This is the direction for possible future investigation.

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