

**CONDORCET CHOICE CORRESPONDENCES
FOR WEAK TOURNAMENTS***

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A B S T R A C T

Tournaments are complete and asymmetric binary relations. This type of binary relation rules out the possibility of ties or indifferences which are common in different contexts. In this work we generalize, from a normative point of view, some important tournaments' solutions (top cycle, uncovered set and minimal covering) to the context where ties are possible.

Keywords: Tournament; Uncovered Set; Minimal Covering.

1. INTRODUCTION

A tournament over a finite set of outcomes A (candidates, decisions,...) is a complete and asymmetric binary relation T on A , where $a T b$ is interpreted as "alternative a beats alternative b ". This kind of binary relations arises in many different models: sport competitions, biometric and psychometric models, collective choice (majority voting rules), ... [see Moon (1968); Moulin (1986)].

If there is an alternative which beats every other, then it is clear that such an alternative must be selected (the winner of the tournament). But this is not the usual case, and in general it is not clear which one (or ones) should be considered the winner of the tournament. Indeed, social choice theorists have constructed several choice rules, but handle the difficulty posed by the nonexistence of a clear winner in different ways.

From a normative point of view, a large number of solutions for the problem of choosing from a tournament has been proposed: Copeland (1951); Slater (1961); top cycle (Schwartz, 1972); uncovered set (Fishburn, 1977; Miller, 1980); minimal covering (Dutta, 1988); equilibrium set (Schwartz, 1990); etc.

Moreover, from a positive point of view, other solution concepts for a tournament have been introduced without characterizing them axiomatically: Banks set (Banks, 1985); bipartisan set (Laffond, Laslier and Le Breton, 1993); matching solutions (Levchenkov, 1994); etc.

As Moulin (1986) points out, "a widely open question is the generalization to any complete relation on A (not necessarily asymmetric: indifferences are allowed)". There is an actual possibility of ties: two football teams can tie; two candidates or alternatives can obtain the same number of votes; ...

The aim of this paper is to generalize, from a normative point of view, some of the previously mentioned solutions for tournaments to the case in which ties are allowed. In particular, the top cycle, the uncovered set and the minimal covering are extended. In some cases this extension is natural, but in other cases there is not a unique clear way to generalize these solution concepts; in this case we have tried to keep the properties (axioms) which are satisfied by the original solutions. Then, we have translated these solutions for complete (not necessarily asymmetric) relations R (**weak tournaments**) in such a way that when R is a tournament the definition coincides with the usual one, and we prove axiomatic characterization results which are quite similar to those of the case of tournaments (in fact, in order to prove some of the results, the techniques used are similar to those in tournaments). Of course, some of the axioms have to be modified, and other new axioms must be introduced.

In the literature there are some papers which analyze what we have called weak-tournaments. In particular, in Banks and Bordes (1988) four extensions of the notion of the uncovered set in tournaments are introduced and axiomatically analyzed. In Henriot (1985) the Copeland set is extended to weak-tournaments and is axiomatically characterized. Later, we will compare these solutions with the ones we will introduce.

The plan of the paper is as follows: in Section 2 some preliminary definitions and properties are introduced. Sections 3, 4 and 5 are devoted to the extensions of the top cycle, uncovered set and minimal covering solutions, respectively. Some final comments close the paper.

2. WEAK TOURNAMENTS AND CHOICE CORRESPONDENCES

A **weak tournament** on A is a pair (R,A) where A is a finite set containing all feasible outcomes, and R is a complete (that is $\forall a,b \in A$, $a R b$ or $b R a$) binary relation on A . From this relation it is always possible to define two new binary relations, P and I , the asymmetric and symmetric part of R , respectively

$$\forall a,b \in A, a P b \text{ if and only if } a R b \text{ and not } (b R a)$$

$$\forall a,b \in A, a I b \text{ if and only if } a R b \text{ and } b R a$$

A **tournament** is the particular case in which

$$\forall a,b \in A, a I b \text{ if and only if } a = b$$

in this case we will represent the asymmetric part of the binary relation by T and we will use the notation (T,A) in order to distinguish this particular case from the general one.

The statement $a P b$ means that alternative a **beats** b in a pair wise comparison, while $a I b$ means a **tie** between both alternatives.

Given a binary relation V defined on A , we will say that an element $a^* \in A$ is V -maximal on A if $a^* V b, \forall b \in A, b \neq a^*$. Given two non-empty subsets of A , we will use the following notation:

$$B V B' \text{ if and only if } b V b' \quad \forall b \in B, \forall b' \in B'$$

$\mathcal{P}(A)$ will denote the set of all non-empty subsets of A ; \mathcal{R} and \mathcal{T} will denote the set of weak tournaments and tournaments on A , respectively.

We will also make use of the **transitive closure** relation: given a binary relation R on A , its transitive closure RR is defined in the following way,

$$\begin{aligned} \forall a, b \in A, a RR b \text{ if and only there are } a_1, a_2, \dots, a_n \in A \\ \text{such that } a = a_1 R a_2 R \dots R a_n = b \end{aligned}$$

This relation depends both on the binary relation R and on the set where it is defined. When necessary, we will denote this dependence by representing the transitive closure of R in $B \subseteq A$ as RR_B .

A choice correspondence S is a mapping

$$S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

such that for every $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$, $S(R, B)$ is a non-empty subset of B . $S(R, B)$ is usually interpreted as the best outcomes of (R, B) .

One of the main problems in choosing the best alternatives from a tournament or weak tournament is that the choice is usually too large (in some cases the whole feasible set). Thus it is usual to look for choice correspondences fulfilling some properties which select as few elements as possible. In this sense, given two choice correspondences S and S' , S is said to be **smaller** than S' (or S' is said to be **larger** than S) if

$$\forall (R,B) \in \mathcal{R} \times \mathcal{P}(A), S(R,B) \subseteq S'(R,B)$$

A choice correspondence S is said to be the smallest (respectively, the largest) satisfying some properties (P), if any other choice correspondence holding (P) is larger (respectively, smaller) than S .

In the case of tournaments, a great number of choice correspondences has been defined, none of them having found universal acceptance. In order to defend and compare different solutions, a multitude of properties have been discussed in the literature. Some of these properties, in the known case of tournaments, are the following:

CONDORCET CONSISTENCY

A choice correspondence $S: \mathcal{T} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies Condorcet consistency if for all $(T,B) \in \mathcal{T} \times \mathcal{P}(A)$,

$$b^* \in B \text{ is } T\text{-maximal in } B \text{ implies } S(T,B) = \{b^*\}$$

Such an element b^* is called a Condorcet-winner.

This condition establishes that if an outcome beats every other element in B , it should be uniquely chosen. Although we only will consider choice correspondences satisfying it, usually named Condorcet choice correspondences, this property gives little information about the solution of a tournament because, in general, there is not a Condorcet-winner. The next two properties are stronger than Condorcet consistency and try to solve the problem of the non-existence of Condorcet-winners.

CONDORCET TRANSITIVITY

A choice correspondence $S: \mathcal{T} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies Condorcet transitivity if for all $(T, B) \in \mathcal{T} \times \mathcal{P}(A)$,

$$a, b \in B, a \in S(T, B), b \notin S(T, B) \text{ implies } a T b$$

In other words, if an outcome beats some chosen outcome, it must be chosen too.

SMITH'S CONDORCET PRINCIPLE

A choice correspondence $S: \mathcal{T} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies Smith's Condorcet principle if for all $(T, B) \in \mathcal{T} \times \mathcal{P}(A)$, $B = B_1 \cup B_2$ and $B_1 T B_2$, then $S(T, B) \subseteq B_1$

This axiom establishes that if we can split the feasible set in two subsets in such a way that every outcome in the first subset beats all the outcomes in the second one, then no outcome in this second subset must be chosen.

In the following definition we translate, in a natural way, these axioms to the context of weak-tournaments.

Definition 1.

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies

A) **Condorcet consistency** if for all $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$,

$$b^* \in B \text{ is P-maximal in } B \text{ implies } S(R, B) = \{b^*\}$$

B) Condorcet transitivity if for all $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$,

$a, b \in B$, $a \in S(R,B)$, $b \notin S(R,B)$ implies $a P b$

C) Smith's Condorcet principle if for all $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$,

$B = B_1 \cup B_2$ and $B_1 P B_2$ implies $S(R,B) \subseteq B_1$

In Banks and Bordes (1988) several conditions, related to the Condorcet consistency, are introduced in the context of weak-tournaments: let $C(R,B)$ the set of R-maximals on A, $C(R,A) = \{x \in A \mid x R y \ \forall y \in A\}$. Then a choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies

Inclusive Condorcet if $C(R,A) \subseteq S(R,A)$

Exclusive Condorcet if $C(R,A) \neq \emptyset$ implies $S(R,A) \subseteq C(R,A)$

Strict Condorcet if $C(R,A) \neq \emptyset$ implies $S(R,A) = C(R,A)$

All of these properties are based on the idea of "weak" Condorcet-winner, that is the existence of an element which beats or ties every other element. The Inclusive and Strict Condorcet principles imply that such elements must be contained in the choice set. But sometimes this fact can be seen as a "bad" property: imagine an outcome, say x , which ties (is indifferent) to every other, and let y be an element which beats every outcome different from x and himself. In this case, it is clear that x may not be chosen as a winner of the weak tournament, contrary to Inclusive and Strict Condorcet properties.

Next we introduce two new axioms; the first one analyzes the choices when we can divide the feasible outcomes into three subsets of indifferent elements with a dominance relation between them; the second one establishes conditions for the choice consisting of exactly two elements.

Definition 2.

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies **Condorcet dominance principle** if for all $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$ such that

$$B = B_1 \cup B_2 \cup B_3; B_i I B_i \forall i; B_i P B_{i+1} \forall i = 1,2; B_1 R B_3$$

then

$$B_{i+1} \cap S(R,B) \neq \emptyset \text{ implies } B_i \cap S(R,B) \neq \emptyset \quad i = 1,2$$

This property establishes that if the subset B_i is ahead of B_{i+1} (in the sense that every outcome in B_i beats any one in B_{i+1}) then if the choice set contains any outcome in B_{i+1} , it should contain someone in B_i . It must be mentioned that, when the relation is a tournament, every Condorcet choice correspondence satisfies this property.

Definition 3.

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies the **binary choice principle** if for all $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$

$$S(R,B) = \{x,y\} \text{ implies } x I y$$

In tournament theory (when indifferences are not allowed), the usual solutions cannot contain exactly two elements (see, for instance, Moulin (1986)). The idea is: if only two elements are chosen and one of them beats the other, it seems "natural" to select only the winner.

Other usual axioms, used in general choice theory, analyze how the choice correspondence changes when the binary relation (tournament or weak tournament) changes.

ARROW'S INDEPENDENCE OF IRRELEVANT ALTERNATIVES

A choice correspondence $S: \mathcal{R} \times \mathcal{B}(A) \longrightarrow \mathcal{B}(A)$ satisfies independence of irrelevant alternatives (AIIA) if for every $B \in \mathcal{B}(A)$ and every $R, R' \in \mathcal{R}$ such that $R|_B = R'|_B$ (that is, R and R' coincide on B) then

$$S(R, B) = S(R', B)$$

NEUTRALITY

A choice correspondence $S: \mathcal{R} \times \mathcal{B}(A) \longrightarrow \mathcal{B}(A)$ is neutral if for any permutation σ of A and every $(R, B) \in \mathcal{R} \times \mathcal{B}(A)$ then

$$S(\sigma(R), B) = \sigma[S(R, B)]$$

where $\sigma(R)$ is the weak tournament defined from R as:

$$\forall a, b \in A, a \sigma(R) b \text{ if and only if } \sigma(a) R \sigma(b)$$

MONOTONICITY

A choice correspondence $S: \mathcal{R} \times \mathcal{B}(A) \longrightarrow \mathcal{B}(A)$ is monotonic if for every $R, R' \in \mathcal{R}$ and every $B \in \mathcal{B}(A)$, such that $R|_{B-\{x\}} = R'|_{B-\{x\}}$ and $\forall b \in B$, $x P b$ implies $x P' b$ and $x I b$ implies $x R' b$, then

$$x \in S(R, B) \text{ implies } x \in S(R', B)$$

Finally, the following axioms (also usual in choice theory) analyze how the choice correspondence changes when the subset presented to choice is modified.

EXPANSION

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies expansion if for any class of sets $\{B_i, i \in I\}$, $B_i \in \mathcal{P}(A) \forall i \in I$ and any $R \in \mathcal{R}$,

$$\bigcap_{i \in I} S(R, B_i) \subseteq S(R, \bigcup_{i \in I} B_i)$$

AXIOM γ^*

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies γ^* if for any class of sets $\{B_i, i \in I\}$, $B_i \in \mathcal{P}(A) \forall i \in I$ and any $R \in \mathcal{R}$,

$$a \in \bigcap_{i \in I} S(R, B_i) \text{ implies } [\bigcup_{i \in I} B_i] - \{a\} \neq S(R, \bigcup_{i \in I} B_i)$$

STRONG SUPERSET PROPERTY

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies the strong superset property (SSP) if for any $B, B' \in \mathcal{P}(A)$ and any $R \in \mathcal{R}$, then

$$S(R, B) \subseteq B' \subseteq B \text{ implies } S(R, B) = S(R, B')$$

AIZERMAN

A choice correspondence $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies Aizerman if for any $B, B' \in \mathcal{P}(A)$ and any $R \in \mathcal{R}$, then

$$S(R, B) \subseteq B' \subseteq B \text{ implies } S(R, B') \subseteq S(R, B)$$

The following result shows how some axioms determine the choice set in some particular weak tournaments.

Lemma 1.

Let $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ a choice correspondence satisfying neutrality and (AIIA). Then,

- 1) If $B = \{a, b, c\}$, $a P b$, $b P c$, $c P a$, then $S(R, B) = B$
- 2) If $a I b$ for every $a, b \in B$, then $S(R, B) = B$

Proof

- 1) See Moulin (1986), page 278.
- 2) Suppose $a \in B$ and $a \notin S(R, B)$. Let $b \in S(R, B)$ and consider the following permutation:

$$\sigma(a) = b, \sigma(b) = a, \sigma(c) = c, \forall c \neq a, c \neq b$$

Neutrality implies $S(\sigma(R), B) = \sigma[S(R, B)]$, so $a \in S(\sigma(R), B)$ and $b \notin S(\sigma(R), B)$. On the other hand, as $\sigma(R)$ coincides with R on B , (AIIA) implies $S(\sigma(R), B) = S(R, B)$, a contradiction. ■

3. THE TOP-CYCLE

One of the basic solution concepts for tournaments is the top cycle (Schwartz, 1972), defined as the set of maximal elements of the transitive closure of T (see Moulin, 1986). In the context of tournaments, the top cycle is the smallest choice correspondence satisfying Condorcet transitivity (Schwartz, 1972) and the largest satisfying Smith's consistency (Moulin, 1986). Moreover, the top cycle is a singleton only when there is a Condorcet winner; otherwise it is a cycle with, at least, 3 elements (Miller, 1977). In the next definition we extend this concept to the context of weak tournaments.

Definition 4.

The top cycle choice correspondence

$$\text{TC}: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

assigns the set of RR-maximal elements on B to each weak tournament $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$,

$$\text{TC}(R, B) = \{a \in B \mid a \text{ RR}_B b, \forall b \in B, b \neq a\}.$$

where, as we have mentioned before, RR_B stands for the transitive closure of relation R on the set $B \in \mathcal{P}(A)$.

The following result shows that the top cycle correspondence, defined in the context of weak tournaments, satisfies the same properties as in the tournament's case.

Theorem 1.

- 1) TC is the smallest choice correspondence satisfying Condorcet transitivity.
- 2) TC is the largest choice correspondence satisfying Smith's Condorcet principle.

Proof

- 1) TC satisfies Condorcet transitivity because if $a \in TC(R,B)$ and $b \in B$, $b R a$, then $b RR_B c$, $\forall c \in B$ and $b \in TC(R,B)$.

Let $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ be a choice correspondence satisfying Condorcet transitivity. If $a \in TC(R,B)$ and $b \in S(R,B)$, then $a RR_B b$, that is,

$$\exists a_1, a_2, \dots, a_n \in B, a = a_1 R a_2 R \dots R a_n = b$$

and applying Condorcet transitivity successively, $a \in S(B,R)$, so

$$TC(B,R) \subseteq S(B,R)$$

- 2) Let $B = B_1 \cup B_2$ such that $B_1 P B_2$. If $a \in B_2 \cap TC(R,B)$ then for all $b \in B_1$ it is verified that $a RR_B b$, that is

$$\exists a_1, a_2, \dots, a_n \in B, a = a_1 R a_2 R \dots R a_n = b$$

Then $a_{n-1} \in B_1$ and successively $a_2 \in B_1$, a contradiction because in this case $a_2 P a$ and $a R a_2$ is not possible.

Let $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ be a choice correspondence satisfying Smith's Condorcet principle. Given $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$, if $a \in S(R, B)$ and $a \notin TC(R, B)$ then there is some $b \in B$, such that $\text{no}(a RR_B b)$ which implies $b P a$ and moreover for all $c \in B$ such that $a RR_B c$, $b P c$. Let

$$B_1 = \{b \in B \mid \text{no}(a RR_B b)\}$$

$$B_2 = B - B_1$$

Then $B_1 P B_2$ and Smith's Condorcet principle implies $S(B, R) \subseteq B_1$, a contradiction because $a \in B_2$. ■

The top cycle is usually a too large choice correspondence. Besides, it has another drawback: it may select Pareto dominated outcomes when the tournament is derived from a binary majority comparison (see Fishburn, 1977 and Moulin, 1986, for comments and examples).

4. THE UNCOVERED SET

In Miller (1977,1980) and Fishburn (1977) the notion of uncovered set is introduced, which is a choice correspondence more discriminating than the top cycle (in particular, in tournaments deduced from majority comparison, the Pareto dominated outcomes do not belong to the uncovered set). An outcome a is in the uncovered set if for every outcome b , a beats b , or a beats some outcome c which in turn beats b (two step principle, Miller, 1977).

Miller (1977) proved that the choice correspondence defined by the uncovered set of a tournament is the smallest satisfying Neutrality, AIIA, Expansion and Condorcet consistency.

In order to extend the definition of the uncovered set to weak-tournaments, we use the following binary relation which is a generalization of the cover relation used by Miller (1977). In the final Section some comments will be made about other ways of generalizing Miller's cover relation.

Definition 5.

Let $(R,B) \in \mathcal{R} \times \mathcal{B}(A)$ be a weak tournament and let $a,b \in B$. It is said that a **R-covers** b **in** B , if and only if

$a P b$, and

$$\forall w \in B, \begin{cases} b P w \text{ implies } a P w \\ b I w \text{ implies } a R w \end{cases}$$

We will denote this fact by $a C_{(R,B)} b$.

It is easy to prove that $C_{(R,B)}$ is a transitive (possibly not complete) binary relation. It is also obvious that when $B' \subseteq B$ and $a, b \in B'$,

$a C_{(R,B)} b$ implies $a C_{(R,B')} b$.

As in the original idea of Miller, when an element b is covered by some other element a , this second element is thought to be better than the first, since a beats b and moreover, a has "better results" than b in a pairwise comparison with the other elements of the alternative set:

if b beats someone, so does a ; if b ties with someone,
 a either beats it or ties with it.

From the covering relation it is possible to define a complete binary relation in the following way:

$a C_{(R,B)}^* b$ if and only if no $(b C_{(R,B)} a)$

As $C_{(R,B)}$ is a transitive relation, $C_{(R,B)}^*$ has maximal elements on every subset B and we can define the following choice correspondence, which generalizes the uncovered set in tournaments.

Definition 6.

The uncovered choice correspondence

$$U: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

assigns the set of all $C_{(R,B)}^*$ -maximal elements on B to each weak tournament $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$

$$U(R,B) = \{a \in B \mid a C_{(R,B)}^* b, \forall b \in B, b \neq a\}$$

It must be noted that if $y \notin U(R,B)$, there is $x \in B$ such that $x \in U(R,B)$ and x R -covers y in B , $x C_{(R,B)} y$. The following result provides a characterization of the uncovered choice correspondence.

Theorem 2.

The uncovered correspondence $U: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is the smallest choice correspondence satisfying (AIIA), Neutrality, Expansion, Aizerman, Condorcet consistency, Condorcet dominance principle and Binary choice principle.

Proof

Let us see first that U satisfies all the properties. It is obvious that (AIIA), Neutrality and Condorcet consistency are fulfilled.

Let $a \in \bigcap_{i \in I} U(R, B_i)$ and suppose $a \notin U(R, \bigcup_{i \in I} B_i)$; let denote $D = \bigcup_{i \in I} B_i$. Then there is some $b \in D$ such that $b C_{(R,D)} a$. But in this case, $b C_{(R, B_j)} a$ for some $j \in I$, a contradiction.

Suppose now $U(R,B) \subseteq B' \subseteq B$ and $a \in U(R, B')$. If $a \notin U(R,B)$ then there exists $b \in U(R,B)$ such that $b C_{(R,B)} a$, but then we obtain that $b C_{(R, B')} a$, which contradicts that $a \in U(R, B')$.

Now let $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$ such that

$$B = B_1 \cup B_2 \cup B_3, B_i \cap B_i \forall i, B_i P B_{i+1} \forall i = 1,2, B_1 R B_3$$

Given $b \in B_1$, there is not $a \in B$ such that $a P b$, so

$$b C_{(R,B)}^* a \quad \forall a \in B$$

and then $B_1 \cap U(R,B) \neq \emptyset$. On the other hand, suppose $B_2 \cap U(R,B) = \emptyset$, and let $c \in B_3$. Then, for every $b \in B_2$, $b P c$ and, as b is not in the uncovering set, there is $a \in B$ such that $a C_{(R,B)} b$ and then $a P c$. It is immediate now that $a C_{(R,B)} c$, so $B_3 \cap U(R,B) = \emptyset$.

Finally, consider (R,B) such that $U(R,B) = \{x,y\}$ and suppose $x P y$. Then, there is $w \in B$ such that one of the following possibilities is fulfilled,

$$y P w R x$$

$$y I w P x$$

In any case, $U(R,B') = \{x,y,w\}$, where $B' = \{x,y,w\}$. Thus, we have

$$U(R,B) \subseteq B' \subseteq B$$

and $U(R,B')$ is not contained in $U(R,B)$, contradicting Aizerman.

To see that the uncovered correspondence is the smallest which satisfies the above mentioned properties, let $S: \mathcal{P} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfying them and let $a \in U(R,B)$. Consider the following partition of set B :

$$B_+ = \{b \in B \mid a P b\}$$

$$B_0 = \{b \in B \mid a I b\}$$

$$B_- = \{b \in B \mid b P a\}$$

Let $b \in B_-$; then, since $a \in U(R,B)$, one of the next three possibilities must occur:

[1] $\exists w_b \in B$ such that $a P w_b$ and $w_b P b$

[2] $\exists w_b \in B$ such that $a P w_b$ and $w_b I b$

[3] $\exists w_b \in B$ such that $a I w_b$ and $w_b P b$

Consider the set $C = \{a, b, w_b\}$. In the first case, Lemma 1 gives us that $S(R, C) = \{a, b, w_b\}$. In case [2], Lemma 1 ensures that $S(R, \{b, w_b\}) = \{b, w_b\}$ and, if $a \notin S(R, C)$, Aizerman implies $S(R, C) = \{b, w_b\}$ contradicting the Condorcet dominance principle. In case [3], the Condorcet dominance principle implies $w_b \in S(R, C)$. If $b \notin S(R, C)$, Lemma 1 and Aizerman imply $S(R, C) = \{a, w_b\}$ and if $b \in S(R, C)$, the binary choice principle implies $S(R, C) \neq \{b, w_b\}$. So, in any case, $a \in S(R, C)$. Then,

$$a \in \bigcap_{b \in B_-} S(R, \{a, b, w_b\})$$

On the other hand, from Lemma 1 and Condorcet consistency

$$a \in \bigcap_{b \in B_+ \cup B_-} S(R, \{a, b\})$$

Finally, Expansion ensures $a \in S(R, B)$. ■

The next property analyzes when the uncovered set contains a unique alternative.

Proposition 1.

$U(R, B) = \{a\}$ if and only if $a C_{(R, B)} b$ for every $b \neq a$

Proof

Consider $b \in B$, $b \neq a$; then as $b \notin U(R,B)$ there is $b_1 \in B$ such that $b_1 C_{(R,B)} b$. If $b_1 \neq a$, by repeating this argument, as the R-cover relation is transitive and B is a finite set, we obtain a $C_{(R,B)} b$. ■

Then as an immediate consequence of the definition of the R-cover relation we obtain:

Corollary 1.

$U(R,B) = \{a\}$ if and only if a is a Condorcet winner

5. THE MINIMAL COVERING

The minimal covering choice correspondence was introduced by Dutta (1988), in the context of tournaments, with the aim of defining a more discriminating solution than the uncovered set. Dutta (1988) proved that, in this context, the minimal covering is the smallest choice correspondence satisfying Neutrality, AIIA, Monotonicity, Strong superset, condition γ^* and Condorcet consistency.

In order to generalize the minimal covering to the context of weak tournaments, first we introduce the translation of the notion of covering set to this case.

Definition 6.

Given $(R,B) \in \mathcal{R} \times \mathcal{L}(A)$, a set $E \subseteq B$ is a covering set of (R,B) if and only if

- a) $U(R,E) = E$
- b) $b \in B - E$ implies $b \notin U(R,E \cup \{b\})$

If E is a covering set of (R,B) , then E is internally stable in accordance with the cover relation defined in the previous Section, since condition a) establishes that all of the elements in E should be uncovered within E . Condition b) requires an external stability in the sense that the elements outside the covering set cannot cover the elements in E .

We denote the family of all covering sets of (R,B) by $\mathcal{C}[R,B]$. The next property relates the covering sets with the uncovered choice correspondence analyzed in the previous Section.

Lemma 2.

If $E \in \mathcal{C}[R,B]$ then $E \subseteq U(R,B)$

Proof

Suppose $b \in E$ and $b \notin U(R,B)$; then there is $c \in B$ such that $c C_{(R,B)} b$ which implies $c C_{(R,E \cup \{c\})} b$. If $c \in E$, a contradiction with the internal stability of the covering set is obtained. Then, $c \notin U(R,E \cup \{c\})$ and so, there is $w \in E$ such that $w C_{(R,E \cup \{c\})} c$. Transitivity of the covering relation gives us $w C_{(R,E \cup \{c\})} b$, which in turn implies $w C_{(R,B)} b$, again a contradiction to internal stability. ■

The following results, Lemma 3 and Theorem 3, are devoted to proving the existence of covering sets for every weak tournament (R,B) .

Consider a weak tournament (R,B) and denote $U^0(R,B) = B$. For any $t \geq 1$, let $U^t(R,B) = U(R, U^{t-1}(R,B))$. As B is a finite set, there is some $k \in \mathbb{N}$ such that $U^k(R,B) = U^{k+1}(R,B)$. We will denote by $U^\infty(R,B)$ the set $U^k(R,B)$ holding this condition.

Lemma 3.

If $b \notin U^\infty(R,B)$, $b \in B$, then there is $a \in U^\infty(R,B)$ such that
 $a C_{(R, U^\infty(R,B) \cup \{b\})} b$

Proof

Let $b \notin U^\infty(R,B)$; then there exists $t \in \mathbb{N}$ such that $b \in U^t(R,B)$ and $b \notin U^{t+1}(R,B)$, that is, there is $c \in U^t(R,B)$ such that $c C_{(R, U^t(R,B))} b$. If

$c \in U^\infty(R,B)$ we have the result, because $U^\infty(R,B) \subseteq U^t(R,B)$. In another case, by following an analogous argument, we can find $d \in U^s(R,B)$, $s \geq t$, such that $d \in C_{(R,U^s(R,B))} c$. As $U^s(R,B) \cup \{b\} \subseteq U^t(R,B)$ for $s \geq t$, and the covering relation is transitive, then $d \in C_{(R,U^s(R,B) \cup \{b\})} b$. By repeating this process, as B is a finite set, we obtain the result. ■

Theorem 3.

$$U^\infty(R,B) \in \mathcal{C}[R,B]$$

Proof

By definition, $U(R,U^\infty(R,B)) = U^\infty(R,B)$. Now, let $c \notin U^\infty(R,B)$; Lemma 3 ensures the existence of $a \in U^\infty(R,B)$ such that $a \in C_{(R,U^\infty(R,B) \cup \{c\})} c$ and then, $c \notin U(R,U^\infty(R,B) \cup \{c\})$. ■

As Lemma 2 showed, every covering set is a selection of the uncovered set. In order to define a choice correspondence for tournaments (or weak tournaments) as discriminating as possible, one way consists of selecting the minimal covering set with respect to the set-inclusion (Dutta, 1988).

Definition 7.

The **minimal covering** choice correspondence

$$MC: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

assigns the minimal (with respect to the set-inclusion) covering set of (R,B) to each weak tournament $(R,B) \in \mathcal{R} \times \mathcal{P}(A)$.

The following results are devoted to proving the existence of such a covering set and that it is unique. So, the minimal covering choice correspondence is well defined.

Lemma 4.

Let $E, F \in \mathcal{C}[R, B]$, then $E \cap F \neq \emptyset$

Proof

Let us suppose $E \cap F = \emptyset$, and consider $e_1 \in E$. As $e_1 \in B - F$, then there is $f_1 \in F$ such that

$$f_1 \ C_{(R, F \cup \{e_1\})} \ e_1$$

As $f_1 \in B - E$, then there is $e_2 \in E$ such that

$$e_2 \ C_{(R, E \cup \{f_1\})} \ f_1$$

In this way, since B is a finite set, we obtain a chain

$$e_{k+1} \ C_{(R, E \cup \{f_k\})} \ f_k \ C_{(R, F \cup \{e_k\})} \ e_k \ C_{(R, E \cup \{f_{k-1}\})} \ f_{k-1} \ \dots \ f_1 \ C_{(R, F \cup \{e_1\})} \ e_1$$

From this, it is immediate that

$$f_t \ P \ e_t \ \text{and} \ e_{t+1} \ P \ f_t, \ t = 1, 2, \dots$$

and then, from the definition of the covering relation,

$$e_{t+1} \ P \ e_t \ \text{and} \ f_{t+1} \ P \ f_t, \ t = 1, 2, \dots \quad [1]$$

As B is a finite set, for a large enough k , $i < j$ must exist in the chain such that $f_i = f_j$ and then

$$f_i \ C_{(R, F \cup \{e_j\})} \ e_j$$

This implies $f_{i+1} \ P \ e_j$ (otherwise $f_i \ R \ f_{i+1}$ contradicting [1]) and, since $e_{i+2} \ C_{(R, E \cup \{f_{i+1}\})} \ f_{i+1}$, $e_{i+2} \ P \ e_j$ which in turn implies $e_{i+2} \ P \ f_{j-1}$ (in another case, $f_{j-1} \ R \ e_{i+2}$ and $e_j \ C_{(R, E \cup \{f_{j-1}\})} \ f_{j-1}$ implies $e_j \ R \ e_{i+2}$, a contradiction) and $f_{i+2} \ P \ f_{j-1}$. By repeating this argument we obtain a contradiction with [1]. ■

Theorem 4.

For all $(R, B) \in \mathcal{R} \times \mathcal{D}(A)$ there is a set $MC(R, B) \in \mathcal{C}[R, B]$ such that for every $E \in \mathcal{C}[R, B]$ then $MC(R, B) \subseteq E$.

Proof

First suppose that there is a Condorcet winner in (R, B) ; then the set containing this element is the unique covering set and the result is true. Note that, when there is not a Condorcet winner, a covering set must contain at least two elements and, in the case where it contains exactly two elements they must be indifferent.

Now consider $(R, B) \in \mathcal{R} \times \mathcal{D}(A)$ and choose a covering M with minimal cardinality. Let $E \subseteq B$ any other covering set of (R, B) . Lemma 4 implies $M \cap E \neq \emptyset$ and then, if M is not contained in E , $M \cap E$ is not a covering set because of the minimal cardinality of M and $U^\infty(R, M \cap E)$ is not a covering set of (R, B) . As $U(R, U^\infty(R, M \cap E)) = U^\infty(R, M \cap E)$ then there is $b \notin U^\infty(R, M \cap E)$ such that

$$b \in U(R, U^\infty(R, (M \cap E) \cup \{b\})) \quad [2]$$

Moreover, as $U^\infty(R, M \cap E) \in \mathcal{C}[R, M \cap E]$, then $b \notin M \cap E$. Without loss of generality suppose $b \notin E$; then, since $b \notin U(R, E \cup \{b\})$, there is $e_1 \in E$ such that $e_1 C_{(R, E \cup \{b\})} b$ and condition [2] implies $e_1 \notin U^\infty(R, M \cap E)$. If $e_1 \in M \cap E$, by Lemma 3, there is $c \in U^\infty(R, M \cap E)$ such that $c C_{(R, U^\infty(R, M \cap E) \cup \{e_1\})} e_1$ and, by transitivity

$$c C_{(R, U^\infty(R, M \cap E) \cup \{b\})} b$$

contradicting [2]. Then $e_1 \notin M \cap E$, so $e_1 \notin M$. As M is a covering set, there is $m_1 \in M$ such that $m_1 C_{(R, M \cup \{e_1\})} e_1$. If $m_1 \in U^\infty(R, M \cap E)$ by transitivity we obtain a contradiction with [2], and by reasoning in a similar way as before, $m_1 \notin M \cap E$, so $m_1 \notin E$. By repeating this argument, we obtain

$$\begin{aligned} m_i &\in M \text{ such that } m_i C_{(R, M \cup \{e_i\})} e_i & i=1,2,\dots \\ e_i &\in E \text{ such that } e_{i+1} C_{(R, E \cup \{m_i\})} m_i & i=1,2,\dots \\ e_1 &\in E \text{ such that } e_1 C_{(R, E \cup \{b\})} b \end{aligned}$$

but this chain of elements is not possible as we have shown in Lemma 4. Therefore we can ensure that $M \subseteq E$. ■

Theorems 5 and 6 provide an axiomatic characterization of the minimal covering choice correspondence.

Theorem 5.

The minimal covering correspondence $MC: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfies Monotonicity, Neutrality, (AIIA), Condorcet consistency, (SSP), condition γ^* and the Condorcet dominance principle.

Proof

It is immediate that MC satisfies Neutrality and (AIIA). In order to prove Monotonicity, let $R, R' \in \mathcal{R}$ and $B \in \mathcal{B}(A)$, such that

$$R|_{B-\{x\}} = R'|_{B-\{x\}}$$

$\forall b \in B, x P b$ implies $x P' b$ and $x I b$ implies $x R' b$, and

$$x \in MC(R, B)$$

Suppose $x \notin MC(R', B)$; we are going to prove that $MC(R', B)$ is a covering set in (R, B) and then $MC(R, B) \subseteq MC(R', B)$, contradicting $x \in M(R, B)$. As the uncovered choice correspondence satisfies (AIIA),

$$U(R, MC(R', B)) = U(R', MC(R', B)) = MC(R', B)$$

Let $b \notin MC(R', B)$. If $b \neq x$, then as $R|_{B-\{x\}} = R'|_{B-\{x\}}$

$$b \notin U(R', MC(R', B) \cup \{b\}) = U(R, MC(R', B) \cup \{b\})$$

If $b = x$, $x \notin U(R', MC(R', B) \cup \{x\})$ and there is $a \in MC(R', B)$ such that

$$a C_{(R', MC(R', B) \cup \{x\})} x$$

that is, $a P' x$, and for every $w \in MC(R', B)$,

$$x P' w \text{ implies } a P' w$$

$$x I' w \text{ implies } a R' w$$

From the relationship between R and R' , it is immediate that a also covers x with the relation R , so $x \notin U(R, MC(R', B) \cup \{x\})$ and then

$$MC(R',B) \in \mathcal{C}[R,B].$$

In order to prove the strong superset property (SSP) consider $B, B' \in \mathcal{B}(A)$ and $R \in \mathcal{R}$, such that $MC(R,B) \subseteq B' \subseteq B$. This implies that $MC(R,B)$ is a covering set in (R',B) and then $MC(R,B') \subseteq MC(R,B)$. Let us suppose $MC(R,B') \neq MC(R,B)$; we are going to prove that, in this case, $MC(R,B')$ is a covering set in (R,B) , contradicting the minimality of $MC(R,B)$.

Obviously, $U(R,MC(R,B')) = MC(R,B')$. On the other hand, if $b \in B - MC(R,B')$ we will distinguish two cases:

1) $b \in B'$; then $b \notin U(R,MC(R,B') \cup \{b\})$, because $MC(R,B')$ is a covering set in (R,B') .

2) $b \notin B'$; in this case, as $MC(R,B) \subseteq B'$, $b \in B - MC(R,B)$ and then, $b \notin U(R,MC(R,B) \cup \{b\})$, so there is $a \in MC(R,B) \subseteq B'$ such that

$$a \in C_{(R,MC(R,B) \cup \{b\})} b$$

If $a \in MC(R,B')$ then $b \notin U(R,MC(R,B') \cup \{b\})$. In another case, $a \in B' - MC(R,B')$ implies the existence of $d \in MC(R,B')$ such that

$$d \in C_{(R,MC(R,B') \cup \{a\})} a$$

and then by transitivity

$$d \in C_{(R,MC(R,B') \cup \{b\})} b$$

So, in any case, $b \notin U(R,MC(R,B') \cup \{b\})$ and $MC(R,B') \in \mathcal{C}[R,B]$.

Now, in order to prove that the minimal covering choice correspondence satisfies condition γ^* , let $B_i \in \mathcal{B}(A)$, $i \in I$, and $R \in \mathcal{R}$, such that

$$a \in \bigcap_{i \in I} MC(R,B_i)$$

and suppose $[\bigcup_{i \in I} B_i] - \{a\} = MC(R, \bigcup_{i \in I} B_i)$; then, there is $b \in MC(R, \bigcup_{i \in I} B_i)$ such that b R -covers a in $MC(R, \bigcup_{i \in I} B_i) \cup \{a\}$. Let $j \in I$ such that $b \in B_j$; then, as $B_j - \{a\} \subseteq MC(R, \bigcup_{i \in I} B_i)$, b R -covers a in B_j which contradicts the fact that $a \in MC(R, B_j)$.

Finally, if $(R, B) \in \mathcal{R} \times \mathcal{B}(A)$, and $B = B_1 \cup B_2 \cup B_3$ such that

$$B_i \cap B_j = \emptyset \quad \forall i, j, \quad B_i \prec B_{i+1} \quad \forall i = 1, 2, \quad B_1 \prec B_3$$

If $B_1 \cap MC(R, B) = \emptyset$, then $MC(R, B) \subseteq B_2 \cup B_3$ and for every $b \in B_1$ there is $c \in B_2 \cup B_3$ such that $c \prec b$; but it is not possible. Now consider (R, B) such that $B_2 \cap MC(R, B) = \emptyset$; then for $a \in B_2$, $a \notin U(R, MC(R, B) \cup \{a\})$ and there is $b \in MC(R, B)$ which R -covers a in $MC(R, B) \cup \{a\}$. In particular, $b \prec a$ which implies $b \in B_1$. If $c \in B_3 \cap MC(R, B)$, then $a \prec c$ and thus b R -covers c in $MC(R, B)$, a contradiction. ■

We are now interested in proving that the minimal covering choice correspondence is more discriminating than any other satisfying the axioms of Theorem 5. First, we prove an auxiliary lemma.

Lemma 5.

Let $(R, B) \in \mathcal{R} \times \mathcal{B}(A)$, and B_1, B_2 be any partition of B such that $B_1 \cap MC(R, B) \neq \emptyset$. Then, there is $a \in B_1$ such that $a \in U(R, B_2 \cup \{a\})$.

Proof

Suppose $a \notin U(R, B_2 \cup \{a\})$ for all $a \in B_1$ and call $B_3 = U^\infty(R, B_2)$. We are going to prove that B_3 is a covering set in (R, B) . From the definition, it is immediate that $U(R, B_3) = B_3$.

Now consider $x \notin B_3$; if $x \in B_1$ there is $b \in B_2$ such that $b \in C_{(R, B_2 \cup \{x\})} x$. If $b \in B_3$ then $b \in C_{(R, B_3 \cup \{x\})} x$; in another case, there is $d \in B_3$ such that $d \in C_{(R, B_3 \cup \{b\})} b$ and the transitivity of the covering relation implies $d \in C_{(R, B_1 \cup \{x\})} x$, so $x \notin U(R, B_3 \cup \{x\})$.

On the other hand, if $x \notin B_1$, then $x \in B_2 - B_3$ and as $B_3 = U^\infty(R, B_2)$, Lemma 3 implies the existence of $b \in U^\infty(R, B_2)$ such that

$$b \in C_{(R, U^\infty(R, B_2) \cup \{x\})} x$$

and then $x \notin U(R, B_3 \cup \{x\})$.

So, $B_3 \in \mathcal{C}[R, B]$ which implies $MC(R, B) \subseteq B_3 \subseteq B_2$, contradicting that B_1, B_2 is a partition of B . ■

Theorem 6.

The minimal covering correspondence $MC: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is the smallest choice correspondence satisfying the axioms of Theorem 5.

Proof

Let $S: \mathcal{R} \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ be a choice correspondence which satisfies all axioms in Theorem 5. If there is $(R, B) \in \mathcal{R} \times \mathcal{P}(A)$ such that

$$MC(R, B) - S(R, B) \neq \emptyset$$

Let $B_1 = MC(R, B) - S(R, B)$ and $B_2 = S(R, B)$. Then,

$$MC(R, B) \subseteq B_1 \cup B_2 \subseteq B$$

and strong superset property implies $MC(R, B) = MC(R, B_1 \cup B_2)$. Lemma 5 thus ensures the existence of $a \in B_1$ such that $a \in U(R, B_2 \cup \{a\})$. Consider the following partition of B_2

$$(B_2)_+ = \{b \in B_2 \text{ such that } a P b\}$$

$$(B_2)_= = \{b \in B_2 \text{ such that } a I b\}$$

$$(B_2)_- = \{b \in B_2 \text{ such that } b P a\}$$

Condorcet consistency implies

$$S(R, (B_2)_+ \cup \{a\}) = \{a\},$$

and from Lemma 1, for every $b \in (B_2)_=$

$$S(R, \{a, b\}) = \{a, b\}$$

On the other hand, as $a \in U(R, B_2 \cup \{a\})$, for every $b \in (B_2)_-$ there is $c \in B_2$ such that one of the following possibilities is fulfilled:

$$(1) \quad a P c P b$$

$$(2) \quad a P c I b$$

$$(3) \quad a I c P b$$

In case (1), Lemma 1 ensures $a \in S(R, \{a, b, c\})$. In case (2), if $a \notin S(R, \{a, b, c\})$, then $S(R, \{a, b, c\}) \subseteq \{b, c\} \subseteq \{a, b, c\}$ and the strong superset property, together with Lemma 1, implies $S(R, \{a, b, c\}) = \{b, c\}$, contradicting the Condorcet dominance principle. So, in case (2), $a \in S(R, \{a, b, c\})$. Finally, in case (3), if $a \notin S(R, \{a, b, c\})$, strong superset and Condorcet consistency imply $S(R, \{a, b, c\}) = \{c\} \subseteq \{a, c\}$, and then, again applying strong superset,

$$S(R, \{a, b, c\}) = S(R, \{a, c\})$$

But $S(R, \{a, c\}) = \{c\}$ contradicts Lemma 1.

Then,

$$a \in S(R, (B_2)_+ \cup \{a\}) \cap \left[\bigcap_{b \in (B_2)_=} S(R, \{a, b\}) \right] \cap \left[\bigcap_{b \in (B_2)_-} S(R, \{a, b, c\}) \right]$$

and condition γ^* implies

$$S(R, B_2 \cup \{a\}) \neq B_2$$

but, $S(R, B) = B_2 \subseteq B_2 \cup \{a\} \subseteq B$, and strong superset implies

$$S(R, B_2 \cup \{a\}) = B_2$$

a contradiction. ■

6. FINAL COMMENTS

It is clear that, as in the case of tournaments, each of the three choice correspondences introduced are more discriminating than the preceding one,

$$\forall (R,B), \quad TC(R,B) \supseteq U(R,B) \supseteq MC(R,B)$$

Moreover, the Copeland-winners (Copeland 1951; Henriot, 1985) are obviously included in the uncovered set and there is no inclusion relation between the Copeland-winners and the minimal covering set. Nevertheless, the Copeland choice correspondence (and, in general, the scoring methods) does not satisfy (AIIA), which is very useful axiom since it implies that the choice on every subset only depends upon the relationship between the outcomes in the subset.

Although the minimal covering is the most discriminating among the solutions of tournaments we have analyzed, as Dutta (1988) points out, the uncovered set has a special appeal: "it is easier to compute and has an interesting interpretation in terms of a real world voting procedure" (sophisticated agenda, see Banks, 1984). The uncovered set (and also the minimal covering) depends on the cover relation which is used in its definition. For tournaments, this relation is defined as

$$a \text{ covers } b \text{ if and only if } a T b \text{ and } a T w \text{ whenever } b T w$$

There is not a unique clear way to translate this definition to the case of weak-tournaments. In Bordes (1986) and Banks and Bordes (1988) two covering relations are examined:

$$a C_u b \text{ if and only if } a P b \text{ and } a R w \text{ whenever } b R w$$

$a C_d b$ if and only if $a P b$ and $a P w$ whenever $b P w$

and from these relations the following uncovered sets can be defined

$$UC_u(R,B) = \{ x \in B \mid y C_u x \text{ for no } y \in B \}$$

$$UC_d(R,B) = \{ x \in B \mid y C_d x \text{ for no } y \in B \}$$

From our definition of the R-cover relation in Section 4, it is immediate to deduce that for every (R,B) ,

a R-covers b in B implies $a C_u b$

a R-covers b in B implies $a C_d b$

and then

$$UC_u(R,B) \cup UC_d(R,B) \subseteq U(R,B)$$

The covering relation we introduced in Section 4 (as well as the cover relation in tournaments) deals with the idea of dominance: an alternative a "dominates" some other alternative b if it has a "better" behavior with respect to any other alternative w in pairwise comparisons. Nevertheless, this is not, in general, the case of the cover relations C_u or C_d since we may have, for instance, the following situations:

$a C_u b$ and $a I w, b P w$ for some w

$a C_d b$ and $w P a, b I w$ for some w

In both situations it is not clear that a has a "better" behavior than b .

On the other hand, each of these covering relation requires a direct strict preference between the two alternatives: that is, if a covers b then $a P b$. A weak dominance relation could only ask for weak preference.

Formally,

$a C_w b$ if and only if $a R b$ and

(1) $b P w$ implies $a P w$

(2) $b I w$ implies $a R w$

(3) there is some $c \in B$ such that

3.1) $a P c, c R b$, or

3.2) $a I c, c P b$

The idea with this new covering relation is to ask for the minimum conditions which preserve the intuition of dominance: it is clear that, if we have two alternatives under the above conditions, the first one must be selected before the second.

Again, it is clear that if $a R$ -covers b , then $a C_w b$ which implies $UC_w(R,B) \subseteq U(R,B)$, where $UC_w(R,B)$ is defined in the usual way. In particular, some alternatives which appear in $U(R,B)$ as well as in $UC_u(R,B)$ and $UC_d(R,B)$, have been eliminated from $UC_w(R,B)$. To clarify this fact, consider the following example:

let $B = \{b_1, b_2, \dots, b_n\}$ and the weak tournament defined as follows,

$b_1 P b_i \quad i = 3, 4, \dots, n$

$b_2 I b_i \quad i = 1, 2, \dots, n$

As b_2 ties with every other alternative, it can not be rejected and belongs to $U(R,B)$ as well as to $UC_u(R,B)$ and $UC_d(R,B)$ (see Theorem 5.2 in Banks and Bordes, 1988). Nevertheless, it seems clear that b_2 can be rejected in front of b_1 . In this example, $UC_w(R,B) = \{b_1\}$.

Note that in the above example there is not a Condorcet winner and the uncovered set $UC_w(R,B)$ only has one element. So, although Proposition 1 is true when we use the cover relation C_w , in this case the result in Corollary 1 can not be obtained.

In order to characterize the elements in $UC_w(R,B)$, we will use the following binary relations obtained from R:

Consider the weak-tournament (R,B)

A) we will say that $a G b$ in B , if there is $c \in B$ such that

(1) $a R c, c P b$, or

(2) $a P c, c R b$

B) we will say that two elements $a,b \in B$ are **indistinguishable** in B , $a \equiv b$, if for any other alternative $w \in B$,

$a P w$ if and only if $b P w$

$a I w$ if and only if $b I w$

$w P a$ if and only if $w P b$

The first relation translates the idea that alternative a beats the alternative b in the feasible set B , either directly or indirectly to the context of weak-tournaments. From these definitions we obtain the following result.

Proposition 2.

$a \in UC_w(R,B)$ if and only if for every $b \in B$

(1) $a G b$, or

(2) $a \equiv b$

Proof

Let $a \in UC_w(R,B)$, then for every $b \in B$, no $b C_w a$. If $a C_w b$ this implies $a G b$. In another case, we may have $a P b$, in which case $a G b$, or $b R a$; but then, as $a \in UC_w(R,B)$, we have the following possibilities:

- (1) there is $c \in B$ such that $a P c$, $c R b$
- (2) there is $c \in B$ such that $a R c$, $c P b$, or
- (3) for every $w \in B$, $a P w$ iff $b P w$
 $a I w$ iff $b I w$

The two first cases imply $a G b$, and the third one shows that $a \equiv b$.

Conversely, consider $a \in B$ such that for every $b \in B$, $a G b$ or $a \equiv b$ and suppose $a \notin UC_w(R,B)$. Then there is $c \in B$ such that $c C_w a$. This implies that a and c can not be indistinguishable, so $a G b$. But this possibility contradicts $c C_w a$. Thus $a \in UC_w(R,B)$. ■

Proposition 2 is a kind of two-step principle (Miller, 1977) for weak tournaments; the difference is that we only ask for a strict preference and a weak preference between the elements and it is only applicable if the elements are not indistinguishable. The next result is an immediate consequence of the definition of indistinguishable alternatives, and it is true for every definition of the uncovered set.

Proposition 3.

$$a \in UC_w(R,B) \text{ and } b \equiv a \text{ implies } b \in UC_w(R,B)$$

Moreover, it is not hard to prove that the choice correspondence defined by $UC_w(R,B)$ satisfies Condorcet-consistency, Neutrality, (AIIA) and Monotonicity. Nevertheless, from an axiomatic point of view, $UC_w(R,B)$ does not keep the properties of the uncovered set in tournaments: Aizerman and Expansion are not satisfied.

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