CLASSICAL EQUILIBRIUM WITH INCREASING RETURNS*

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A B S T R A C T

This paper analyzes the existence of equilibrium in a market economy with increasing returns to scale. Consumers and firms are modelled as payoff maximizers at given prices within their feasible sets. Firms are to be thought of as created by a set of consumers willing to operate some of the available technological possibilities while, at the same time, providing the required means to enable this. Rational consumers will only be willing to set up firms if they can achieve the maximum profitability attainable. A Classical Equilibrium consists of a price vector and an allocation such that supply equals demand, all active firms are equally profitable, and this is the maximum profitability attainable at given prices.

Keywords: Classical Equilibrium; Increasing Return; Distributive Sets.

1.- INTRODUCTION

This paper presents a contribution to the existence of market equilibria in economies which may exhibit increasing returns to scale [the reader is referred to Brown (1991), Quinzii (1992) or Villar (1994) for a review of the literature in this field]. The model developed is positive rather than normative, and refers to a static market economy with competitive features. The key concept in the analysis will be that of a Classical Equilibrium. A Classical Equilibrium consists of a price vector and an allocation such that supply equals demand, and all active firms are equally profitable (where the common rate of return is the highest one attainable at these prices). We shall concentrate on the case in which the maximum profitability is zero (what will be called Canonical Classical Equilibrium).

The idea that competition is a process which implies the equalization of firms' profitability is an old one. It played a central role in the modelling of markets by classical economists, such as Smith, Ricardo or Marx (but also Walras, Wicksell or Hayek). This is an appealing concept which reflects the combination of three key attributes of competitive markets: (1) Technology is freely available (that is, there are no "barriers to entry"); (2) Production and exchange are voluntary (that is, no agent can be forced to participate into production and exchange); and (3) Prices are outside the control of individual agents (which can be identified with price-taking behaviour). As a consequence, in a private ownership competitive economy, no agent will willingly accept a smaller return from her "investment" than the highest one attainable at given prices, so that firms will only become active in those activities which yield such a profitability. Note that these ideas are relatively independent of the nature of technology: no matter the kind of returns to scale prevailing, as far as the aforementioned properties hold, the classical notion of competition can be applied. The model presented here will be based on this fact.

The main features of the single-period private ownership market economy we shall be referring to are the following:

(i) Commodities will be divided into two groups: (a) Produced Commodities, which correspond to those natural resources (to be interpreted as "produced by nature"), factors of production and consumption goods produced "yesterday" and/or inherited from the past; they constitute the initial endowments of today's economy. And (b) New Goods, which include both consumption goods, and other inputs which can be produced today.

(ii) The technology will be modelled in terms of a finite number of production sets, which describe different production activities or economic sectors. These activities exhibit non-decreasing returns to scale; in particular, it will be assumed that production sets are distributive [Scarf (1986)]. There is free access to the technology.

(iii) Consumers are characterized by their consumption sets, their utility functions and their initial endowments, and are standard concerning these respects. They choose consumption vectors in order to maximize utility at given prices, subject to their budget constraints.

Consumers' decisions also refer to the use of their initial holdings for production purposes (they contribute to the creation of firms by making available their endowments, looking for the highest profitability of their "investment").

(iv) Firms are not given a priori, but appear as part of consumers' optimal decisions. A firm is created when a set of consumers coordinate on the use of some of the technological possibilities, by providing the factors that might be required. Firms maximize profits at given prices, subject to their feasible sets (i.e., subject to the amounts of inputs provided by consumers at market prices).

Observe that a key characteristic of the way of modelling the economy is the distinction between technology (which belongs to the data of the model) and firms (which are dependent on consumers' decisions). The underlying idea is that factors have to be made available before production takes place. This implies, on the one hand, that a firm does not exist unless consumers provide some factors. And, on the other, that firms will be characterized by both the nature of production activities they carry out, and their feasible sets.

The model may well be interpreted as a two-stage process. In the first stage consumers take investment decisions and firms are created. In the second stage production takes place, consumers get paid and demand is realized. Yet, for the sake of simplicity in exposition, the model refers to a single-period economy. This feature also permits one to discuss the role of profits independently of the "interest rate" (or "discount factor").

The paper is organized as follows. Section 2 presents the economy under consideration. Section 3 is devoted to the analysis of Canonical Classical Equilibria. It is shown that a Canonical Classical Equilibrium exists and it is efficient conditional on the allocation of Produced Commodities; a particular case where some equilibrium allocations are in the core is also analyzed. The case of Classical Equilibria with positive profit rates is discussed in section 4. A few final comments in Section 5 close the paper. The proofs of the theorems will be relegated to an Appendix.

2.- THE MODEL

Consider a single-period, private ownership market economy, with ℓ commodities. Commodities 1, 2, ..., k are New Goods, while commodities k+1, k+2, ..., ℓ are Produced Commodities. New Goods are consumption goods and inputs that can be produced today. Produced Commodities are consumption goods and inputs to production (factors) which are inherited from the past. These commodities cannot be produced again[1], and hence limit today's production possibilities. A point $\omega \in \mathbb{R}^{\ell}$ denotes the aggregate vector of initial endowments. According to the previous classification, this vector takes the form $\omega = (0, \sigma)$, where $0 \in \mathbb{R}^{k}$, and σ is a point in $\mathbb{R}^{\ell-k}$.

Production possibilities are described by means of n production sets. Each of these sets summarizes the technical knowledge of a specific production activity. Production activities differ in the kind of inputs they use and/or the type of outputs they obtain. One may well interpret these activities as the "industries" or "sectors" of an economy. Thus, for j = 1, 2, ..., n, $Y_j \subset \mathbb{R}^{\ell}$ stands for the jth production set, while a point $\mathbf{y}_j \in Y_j$ denotes a production plan which is feasible for the jth activity. $\mathfrak{F}_j$ stands for the jth set of weakly efficient production plans, that is[2],

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[1] Let us recall here that two commodities which are identical physically will be considered as different commodities if they are produced at two different periods.

[2] The convention for vector comparisons is: ≥, >, >>.

$$\mathfrak{F}_j \equiv \{ \mathbf{y}_j \in Y_j \ / \ \mathbf{y}'_j >> \mathbf{y}_j \implies \mathbf{y}'_j \notin Y_j \}$$

A point $\mathcal{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ denotes an element of $\prod_{j=1}^{n} \mathfrak{F}_j$.

According to the classification of commodities above, a production plan for the *jth* activity can be written as: $\mathbf{y}_j = (\mathbf{b}_j, \mathbf{a}_j)$, with $\mathbf{b}_j \in \mathbb{R}^k$ and $\mathbf{a}_j \in -\mathbb{R}_+^{\ell-k}$. No sign restriction is established on $\mathbf{b}_j$, so that there may be New Goods used as inputs in today's production. The technology (which encompasses all activities) is publicly known and freely accessible.

A point $\mathbf{p} \in \mathbb{R}_+^{\ell}$ denotes a price vector. The scalar product $\mathbf{py}_j$ for $\mathbf{y}_j$ in Y_j gives us the profits associated with $\mathbf{y}_j$ at prices $\mathbf{p}$.

Let $r_j: \mathbb{R}_+^{\ell} \times \mathfrak{F}_j \longrightarrow \mathbb{R}$ be a mapping given by:

$$r_j(\mathbf{p}, \mathbf{y}_j) \equiv \begin{cases} \mathbf{py}_j / \mathbf{p}(0, -\mathbf{a}_j), & \text{if } \mathbf{y}_j \neq \mathbf{0} \\ 0, & \text{otherwise} \end{cases}$$

r_j is left undefined when $\mathbf{p}(0, -\mathbf{a}_j) = 0$ and $\mathbf{y}_j \neq \mathbf{0}$. Then, call $r(\mathbf{p}, \mathcal{Y})$ to the maximum profitability attainable (whenever defined), that is:

$$r(\mathbf{p}, \mathcal{Y}) \equiv \underset{j}{\text{Max.}} \{ r_j(\mathbf{p}, \mathbf{y}_j) \}$$

Hence, $r(\mathbf{p}, \mathcal{Y})$ tells us the biggest return one can get per dollar invested, when prices are $\mathbf{p}$ and production is evaluated at $\mathcal{Y}$.

Let $\rho \in \mathbb{R}_+$ be a scalar, to be interpreted as a parametric rate of profits, and consider the following definitions which will enable to make precise (and non-vacuous) the equilibrium notion:

Definition 1.- The pair $(\mathbf{p}', \mathbf{y}'_j) \in \mathbb{R}^{\ell}_+ \times \mathfrak{Y}_j$, is an **equilibrium relative to ρ for the *jth* activity**, if:

(i) For $\mathbf{y}'_j \neq 0$,

$$\mathbf{p}'(0, -\mathbf{a}'_j)\rho = \mathbf{p}'\mathbf{y}'_j \geq \mathbf{p}'\mathbf{y}_j \quad \forall\ \mathbf{y}_j \in Y_j \quad \text{such that } \mathbf{a}_j \geq \mathbf{a}'_j$$

(ii) For $\mathbf{y}'_j = 0$, $\mathbf{p}'$ belongs to the closed convex hull of the following set:

$$\{\ \mathbf{q} \in \mathbb{R}^{\ell}_+ \ /\ \exists \{\mathbf{q}^{\nu}, \mathbf{y}^{\nu}_j\} \subset \mathbb{R}^{\ell}_+ \times [\mathfrak{Y}_j \setminus \{0\}]$$
$$\text{with } \{\mathbf{q}^{\nu}, \mathbf{y}^{\nu}_j\} \longrightarrow (\mathbf{q}, 0) \text{ and } \mathbf{q}^{\nu}\mathbf{y}^{\nu}_j = \mathbf{q}^{\nu}(0, \mathbf{a}^{\nu}_j)\rho\ \}$$

Definition 2.- A pair $(\mathbf{p}', \mathcal{Y}') \in \mathbb{R}^{\ell}_+ \times \prod_{j=1}^{n} \mathfrak{Y}_j$ is a **Production Equilibrium relative to ρ** if, for all $j = 1, 2, \ldots, n$, $(\mathbf{p}', \mathbf{y}'_j)$ is an equilibrium relative to ρ for the *jth* activity.

The *jth* activity is in equilibrium relative to ρ when $\mathbf{y}'_j = (\mathbf{b}'_j, \mathbf{a}'_j)$ is a profit maximizing production plan at prices $\mathbf{p}'$, subject to the restriction of not using more Produced Commodities than those determined by $\mathbf{a}'_j$, and such that $\mathbf{p}'\mathbf{y}'_j = \mathbf{p}'(0, -\mathbf{a}'_j)\rho$. Since this places no restriction on prices when $\mathbf{y}'_j = 0$, we require in this case that $(\mathbf{p}', 0)$ is a limit point of a sequence of pairs yielding a profitability equal to ρ. A Production Equilibrium is a situation where all activities are in equilibrium relative to ρ, for the same price vector.

There are m consumers, which are supposed to behave <u>competitively</u>. Each consumer $i = 1, 2, \ldots, m$, is characterized by a collection

$$[X_i,\ u_i,\ \omega_i,\ W_i]$$

where $X_i \subset \mathbb{R}^{\ell}$, $u_i: X_i \longrightarrow \mathbb{R}$, and $\omega_i = (0, \sigma_i) \in \mathbb{R}^{\ell}$ stand for the *ith* consumer's consumption set, utility function and initial endowments, respectively, and $W_i: \mathbb{R}^{\ell}_{+} \times \prod_{j=1}^{n} \mathfrak{F}_j \longrightarrow \mathbb{R}$ denotes the *ith* consumer's wealth function.

Consumers own the initial endowments and maximize utility at given prices, by suitably choosing consumption bundles under the restriction of their available wealth. Wealth is given by the exchange value of their initial endowments, which depends upon prices and the profits that can be obtained by applying their resources to production activities. By construction, consumers are not interested in the nature of production activities they support, but just in the profitability they can obtain. Thus, whenever any two activities yield the highest profitability attainable, they will be indifferent in applying their resources to any of them. Moreover, the supply of inputs will only be directed to those activities yielding a return equal to $r(\mathbf{p}, \mathcal{Y}) \geq 0$.

This can formally be expressed as follows. Let $\alpha_{ij}(\mathbf{p}, \mathcal{Y}) \in \mathbb{R}^{\ell-k}$ denote the *ith* consumer's investment in the *jth* sector, and let $\alpha_i(\mathbf{p}, \mathcal{Y})$ in $\mathbb{R}^{(\ell-k)n}$ stand for the *ith* consumer's overall investment distribution. For any given pair $(\mathbf{p}, \mathcal{Y})$, $\alpha_i(\mathbf{p}, \mathcal{Y})$ solves the program:

$$\text{Max } W_i(\mathbf{p}, \mathcal{Y}) \equiv \sum_{j=1}^{n} \mathbf{p}(0, \alpha_{ij})[1 + \max\{0, r_j(\mathbf{p}, \mathbf{y}_j)\}]$$

subject to:

$$\sum_{j=1}^{n} \alpha_{ij} \leq \sigma_i$$

For a given $(\mathbf{p}, \mathcal{Y})$, $I_i(\mathbf{p}, \mathcal{Y})$ stands for the *ith* consumer's aggregate supply of inputs, that is, the set of points $\mathbf{t} \in \mathbb{R}^{\ell-k}$ such that $\mathbf{t} = \sum_{j=1}^{n} \alpha_{ij}(\mathbf{p}, \mathcal{Y})$. Note that this supply of inputs can be thought of as a correspondence I_i from $\mathbb{R}_+^{\ell} \times \prod_{j=1}^{n} \mathfrak{F}_j$ into $\mathbb{R}^{\ell-k}$, such that:

$$I_i(\mathbf{p}, \mathcal{Y}) = \{ 0 \}, \quad \text{if } r(\mathbf{p}, \mathcal{Y}) < 0$$

$$I_i(\mathbf{p}, \mathcal{Y}) = [0, \sigma_i], \quad \text{if } r(\mathbf{p}, \mathcal{Y}) = 0$$

$$I_i(\mathbf{p}, \mathcal{Y}) = \{ \sigma_i \}, \quad \text{if } r(\mathbf{p}, \mathcal{Y}) > 0$$

Since no agent can be forced to participate into production, whenever $r(\mathbf{p}, \mathcal{Y})$ is negative, consumers will not develop production activities. When $r(\mathbf{p}, \mathcal{Y}) = 0$ the *ith* consumer's wealth is given by $\mathbf{p}(0, \sigma_i)$, no matter how she allocates her initial endowments; hence the supply of inputs can be taken as the whole interval $[0, \sigma_i]$. Finally, if $r(\mathbf{p}, \mathcal{Y}) > 0$ the *ith* consumer will be willing to devote all her resources to the most profitable activities, because this maximizes her wealth (let us recall here that consumers behave competitively, so that they make choices without taking into account any restriction other than wealth).

This allows us to express the *ith* consumer's wealth function as follows:

$$W_i(\mathbf{p}, \mathcal{Y}) \equiv \mathbf{p}\omega_i[1 + \max. \{ 0, r(\mathbf{p}, \mathcal{Y}) \}]$$

[which may not be defined for some pairs $(\mathbf{p}, \mathcal{Y})$].

Remark 1.- Note that this formulation implies, for $r(\mathbf{p}, \psi) > 0$, that consumers cannot trade with "future yields" before production takes place. For interpretative purposes we can think that consumers "invest" at the beginning of the period, then production takes place, and finally consumers get paid and actually consume at the end of the period. We shall refer again to this question in the final section.

The *ith* consumer's demand obtains as a solution to the following program:

$$\begin{aligned} & \text{Max. } u_i(\mathbf{x}_i) \\ \text{s.t.:} \quad & \mathbf{x}_i \in X_i \\ & \mathbf{p}\,\mathbf{x}_i \leq W_i(\mathbf{p}, \psi) \end{aligned}$$

Then, the *ith* consumer's behaviour can be summarized by a demand correspondence $\xi_i : \mathbb{R}^{\ell}_{+} \times \prod_{j=1}^{n} \mathfrak{F}_j \longrightarrow X_i$, such that $\xi_i(\mathbf{p}, \psi)$ stands for a solution to the program above.

We have already described the technology and the consumption sector. Let us now refer to firms. A firm results from the application of resources to put into work some of the possibilities that the available technology offers. These resources have to be made available "before" production takes place(3). Thus a firm can be described by the nature of its production activities and its feasible set (given by the amounts of factors available).

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3 This is an intuitive way of saying that Produced Commodities are essential inputs to production. The assumptions will make clear this point.

For every pair $(\mathbf{p}', \mathcal{y}') \in \mathbb{R}^{\ell}_{+} \times \prod_{j=1}^{n} \mathfrak{Y}_j$, consumers will decide to set up firms by choosing the most profitable production activities and making available the inputs they own. In order to make things simpler, let us assume that there can be at most one firm per activity[4], and consider the following definition:

Definition 3.- Given a point $(\mathbf{p}', \mathcal{y}') \in \mathbb{R}^{\ell}_{+} \times \prod_{j=1}^{n} \mathfrak{Y}_j$, an **Input Allocation relative to** $(\mathbf{p}', \mathcal{y}')$ is a point $a \equiv (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ in $\mathbb{R}^{(\ell-k)n}_{+}$, such that, for every $j = 1, 2, \dots, n$, one has:

$$\mathbf{a}_j = \sum_{i=1}^{m} \alpha_{ij}(\mathbf{p}, \mathcal{y})$$

An Input Allocation relative to $(\mathbf{p}', \mathcal{y}')$ is a way of allotting Produced Commodities to firms which is consistent with consumers' decisions. In particular, this implies that no firm will be created in those activities such that $r_j(\mathbf{p}', \mathbf{y}'_j)$ is smaller than $r(\mathbf{p}', \mathcal{y}')$ (that is, $r_j(\mathbf{p}', \mathbf{y}'_j) < r(\mathbf{p}', \mathcal{y}')$ implies $\mathbf{a}_j = 0$), and that $\sum_{j=1}^{n} \mathbf{a}_j \in \sum_{i=1}^{m} I_i(\mathbf{p}, \mathcal{y})$. Needless to say that there are many Input Allocations relative to a given pair $(\mathbf{p}', \mathcal{y}')$, and that all of them are equally worthy from consumers' viewpoint.

Thus, given a price vector $\mathbf{p}' \in \mathbb{R}^{\ell}_{+}$, a vector of production plans $\mathcal{y}'$ in $\prod_{j=1}^{n} \mathfrak{Y}_j$, and an Input Allocation a relative to $(\mathbf{p}', \mathcal{y}')$, the *jth* firm's

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4 It will be seen soon after that this implies no loss of generality, under the assumptions of our model.

feasible set is given by:

$$Y_j(\mathbf{p}', \mathcal{Y}', a) \equiv \{ \mathbf{y}''_j \in Y_j \ / \ \mathbf{a}''_j \leq \mathbf{a}_j \}$$

The behaviour of firms can now be described as follows: Given a point $(\mathbf{p}', \mathcal{Y}') \in \mathbb{R}^{\ell}_{+} \times \prod_{j=1}^{n} \mathfrak{F}_j$ and an Input Allocation a relative to $(\mathbf{p}', \mathcal{Y}')$, the *jth* firm's supply is obtained by solving the program:

$$\begin{aligned} &\text{Max. } \mathbf{p}'\mathbf{y}''_j \\ &\text{s.t.} \\ &\quad \mathbf{y}''_j \in Y_j(\mathbf{p}', \mathcal{Y}', a) \end{aligned}$$

that is, the *jth* firm maximizes profits over its feasible set.

Consider now the following definitions:

Definition 4.- A **Classical Equilibrium relative to** ρ is a price vector $\mathbf{p}^* \in \mathbb{R}^{\ell}_{+}$, and an allocation $[(\mathbf{x}^*_i), \mathcal{Y}^*] \in \prod_{i=1}^{m} X_i \times \prod_{j=1}^{n} \mathfrak{F}_j$, such that:

(α) $\mathbf{x}^*_i \in \xi_i(\mathbf{p}^*, \mathcal{Y}^*)$, $\forall\ i$.

(β) $a^* \equiv (\mathbf{a}^*_1, \mathbf{a}^*_2, \ldots, \mathbf{a}^*_n)$ is an Input Allocation relative to $(\mathbf{p}^*, \mathcal{Y}^*)$ [where these $\mathbf{a}^*_j$ are such that $\mathbf{y}^*_j = (\mathbf{b}^*_j, \mathbf{a}^*_j)$].

(γ) $(\mathbf{p}^*, \mathcal{Y}^*)$ is a Production Equilibrium relative to $\rho = r(\mathbf{p}^*, \mathcal{Y}^*)$.

(δ) $\sum_{i=1}^{m} \mathbf{x}^*_i = \sum_{j=1}^{n} \mathbf{y}^*_j + \omega$

Definition 5.- A **Canonical Classical Equilibrium** is a Classical Equilibrium relative to $\rho = 0$.

That is, a Classical Equilibrium is a price vector and an allocation such that: (a) Consumers maximize preferences within their budget sets; (b) Consumers voluntarily provide all inputs which are needed for production purposes; (c,1) All firms maximize profits at given prices subject to their feasible sets; (c,2) All active firms are equally profitable; (c,3) The common rate of return is the maximum profitability attainable across sectors; and (d) All markets clear. In the case of a Canonical Classical Equilibrium, it is also true that the common rate of return is $r(\mathbf{p}^*, \mathcal{y}^*) = 0$. Observe that part ($\gamma$) of the definition implies that we are discarding those trivial equilibria obtained by setting $\mathbf{y}_j = \mathbf{0}$ for all j and finding a pure exchange equilibrium.

The next definition singularizes a class of non-decreasing returns to scale production sets which will show most useful:

Definition 6.- [Scarf (1986)] A production set Y_j is said to be **distributive,** if for any collection of points $(\mathbf{y}^t, \lambda^t)$, $t = 1,2, \ldots, s$, with $\mathbf{y}^t = (\mathbf{b}^t, -\mathbf{a}^t) \in Y_j$, $\lambda^t \in \mathbb{R}_+$, the following condition holds:

$$\sum_{h=1}^{s} \lambda^h \mathbf{a}^h \geq \mathbf{a}^t,\ t = 1, 2, \ldots, s \Longrightarrow \sum_{h=1}^{s} \lambda^t \mathbf{y}^t \in Y_j$$

In words: A production set is distributive when any nonnegative weighted sum of feasible production plans is feasible, if it does not use fewer inputs than any of the original plans. From a geometrical standpoint, this amounts to saying that a straight line connecting any point $\mathbf{y}'_j$ in the boundary of Y_j with the origin, does not cut the interior of the set $\{ \mathbf{y}_j \in Y_j \ / \ \mathbf{a}_j \geq \mathbf{a}'_j \}$. It can be seen that if a production set is distributive, then it exhibits non-decreasing returns to scale, and has convex iso-inputs sets [that is, the set $B(\mathbf{a}_j) \equiv \{\mathbf{b}_j \in \mathbb{R}^k \ / \ (\mathbf{b}_j, \mathbf{a}_j) \in Y_j\}$ is convex]. Distributivity ensures not only the additivity of production sets, but also that the constrained profit maximization process which characterizes the behaviour of firms, is well defined and compatible with zero profits (indeed this property practically characterizes those production sets for which average cost pricing and input-constrained profit maximization are compatible). See the discussion in Scarf (1986), Dehez & Drèze (1988 b) and Quinzii (1992, Ch. 6).

Let $\mathcal{A}$ stand for the set of attainable allocations, that is,

$$\mathcal{A} \equiv \{ [(\mathbf{x}_i), \mathcal{y}] \in \prod_{i=1}^{m} X_i \times \prod_{j=1}^{n} \mathfrak{F}_j \ / \ \sum_{i=1}^{m} \mathbf{x}_i \leq \omega + \sum_{j=1}^{n} \mathbf{y}_j \}$$

The projection of $\mathcal{A}$ over the *jth* production set gives us the set of attainable production plans for the *jth* activity.

In order to get a suitable bound for the rate of profits, for each ρ in $\mathbb{R}_+$, define a set $\mathcal{A}(\rho)$ as follows:

$$\mathcal{A}(\rho) \equiv \{ [(\mathbf{x}_i), \mathcal{y}] \in \prod_{i=1}^{m} X_i \times \prod_{j=1}^{n} \mathfrak{F}_j \ / \ \sum_{i=1}^{m} \mathbf{x}_i - \omega - \sum_{j=1}^{n} [\mathbf{b}_j, \mathbf{a}_j(1+\rho)] \leq \mathbf{0} \}$$

$\mathcal{R}$ will denote the set of values of $\rho \in \mathbb{R}_+$ for which $\mathcal{A}(\rho)$ is nonempty. Note that this set is an interval which will be nonempty whenever the set of attainable allocations be nonempty.

We are already prepared to present the basic assumptions of our model:

A.1.- For each consumer $i = 1, 2, ..., m$,

(i) $X_i \subset \mathbb{R}^\ell$ is a closed and convex set, bounded from below.

(ii) $u_i : X_i \longrightarrow \mathbb{R}$ is a continuous and quasi-concave utility function, satisfying local non-satiation.

A.2.- For each $j = 1, 2, ..., n$,

(i) Y_j is a closed set such that $\mathbf{0} \in Y_j$, and $Y_j - \mathbb{R}^\ell_+ \subset Y_j$.

(ii) The *jth* firm's attainable production set is compact. In particular, $\mathbf{a}_j = \mathbf{0}$ implies $\mathbf{b}_j \leq 0$.

(iii) Y_j is distributive.

A.3.- Let $(\mathbf{p}', \mathcal{y}')$ be a Production Equilibrium relative to ρ, for $\rho \in \mathcal{R}$. Then for every $i = 1, 2, ..., m$,

$$W_i(\mathbf{p}', \mathcal{y}') > \text{Min. } \mathbf{p}\mathbf{x}_i, \text{ for } \mathbf{x}_i \text{ in } X_i.$$

Assumption (A.1) refers to consumers. It is assumed there that each individual has a complete, continuous and convex preference preordering, defined over a closed and convex set, and satisfying a non-satiation property.

Assumption (A.2) describes the basic features of technology. Besides closedness, it is assumed in Part (i) that inactivity is possible and that there is free disposal. Part (ii) directly assumes the compactness of attainable production sets; it also says that positive production requires using up some Produced Commodities (this translates the idea that inputs have to be made available "before" production takes place). Finally, Part (iii) assumes the distributivity of production sets (see the explanation above); this property implies that all firms exhibit non-decreasing returns to scale.

Assumption (A.3) says that, in a production equilibrium relative to $\rho \in \mathcal{R}$, every consumer satisfies the "cheaper point" requirement (that is, all consumers' budget sets have a nonempty interior).

Let us conclude this section by discussing the simplifying hypothesis of (at most) one active firm per activity. Suppose, for the sake of the argument, that any number of firms can be created in a given sector. Note that, under assumption (A.2), there can only be production activities with non-decreasing returns to scale. For those sectors exhibiting constant returns to scale, the number of firms is actually irrelevant (since production sets are convex cones, and thus satisfy additivity and divisibility). As for those sectors with increasing returns to scale, observe that, in equilibrium, there can only be one active firm in each activity. Otherwise $(\mathbf{p}^*, \mathcal{y}^*)$ would not be a Production Equilibrium relative to $\rho = r(\mathbf{p}^*, \mathcal{y}^*)$. Therefore, modelling one firm per activity has only served the purpose of simplifying the writing of the model, without any loss of generality.

3.- CANONICAL CLASSICAL EQUILIBRIUM

This section refers to Canonical Classical Equilibria (i.e., the case where $r(\mathbf{p}^*, \mathcal{Y}^*) = 0$), while the case of Classical Equilibrium with positive profits will be analyzed in the next one.

Let us start by presenting the main result of this section:

Theorem 1.- Let E be an economy satisfying assumptions (A.1) to (A.3). Then:

(i) A Canonical Classical Equilibrium $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathcal{Y}^*]$ exists.

(ii) There is no feasible allocation $[(\mathbf{x}'_i), \mathcal{Y}']$, such that $u_i(\mathbf{x}'_i) \geq u_i(\mathbf{x}_i^*)$, $\forall\ i$, with at least a strict inequality, and $\mathbf{a}'_j \geq \mathbf{a}_j^*$ for all j.

Theorem 1 tells us that there exist Canonical Classical Equilibria under fairly general assumptions. These equilibria may be regarded as describing a market situation where production is carried out, based on consumers' rational behaviour, in order to exploit the benefits derived from technical knowledge. Consumers maximize utility subject to their budget constraints. Firms are created only when they yield the highest profitability attainable at given prices, and maximize profits subject to their feasible sets. Theorem 1 ensures that all these actions are compatible for some pair $(\mathbf{p}^*, \mathcal{Y}^*)$. It also establishes that there is no feasible way of making consumers better-off, preserving the distribution of Produced Commodities between firms. Even though this is a mild efficiency property, let us recall here that, in the presence of increasing returns, equilibrium allocations typically fail to satisfy Pareto optimality [see for instance Vohra (1991)].

Remark 2.- It is interesting to note the analogy between the notions of Canonical Classical Equilibrium and Scarf's (1986) Social Equilibrium. Indeed, the former can partly be seen as rationalizing the latter, when we approach the equilibrium problem from a positive viewpoint. This is of some import, since most of the literature dealing with general equilibrium with increasing returns follows a normative approach [see however Dehez & Drèze (1988 a, b)].

Let $N = \{ 1, 2, ..., n \}$ denote the set of indices identifying the firms of the economy. Assuming that (iii) of (A.2) holds, these firms can be divided into two categories: firms with constant returns to scale, and firms with increasing returns to scale. One can thus write $N = N_0 \cup N_1$, where N_0 is the set of indices corresponding to constant returns to scale firms, and N_1 its complement.

The following Corollary is an immediate consequence of Theorem 1:

Corollary 1.- Let $[p^*, (x_i^*), y^*]$ be a Canonical Classical Equilibrium. Then, those firms in N_0 behave as (unconstrained) profit maximizers at given prices.

Thus a Canonical Classical Equilibrium gives us a picture of an economy where constant returns to scale firms behave as unconstrained profit maximizers, while natural monopolies maximize profits subject to their feasible sets, and all firms do break even. All these actions are compatible with consumption and "investment" decisions. If N is empty, then all firms behave as (unconstrained) profit maximizers at given prices and the notion of Canonical Classical Equilibrium coincides with the standard one.

Consider now the following assumption, which restricts the model to a case where Produced Commodities are not consumed:

A.4.- Produced Commodities $h = k+1, k+2, \ldots, \ell$, are pure inputs, so that they do not enter the preferences of consumers.

The following result can be obtained:

Theorem 2.- Let E be an economy satisfying assumptions (A.1), (i) and (ii) of (A.2), (A.3) and (A.4). Suppose furthermore that $Y_0 = \sum_{j=1}^{n} Y_j$ is distributive. Then there exists a Canonical Classical Equilibrium which is in the core.

This Theorem provides us with sufficient conditions for the efficiency and social stability of some equilibrium allocations. Note that the Theorem says that there exists a Canonical Classical Equilibrium which is in the core, not that every equilibrium allocation is a core allocation. Yet, the possibility that agents coordinate on an inefficient equilibrium lacks of social stability.

Remark 3.- Scarf's (1986) main result shows (when reinterpreted in our context) that if n = 1, assumptions (A.1) to (A.4) imply that equilibrium allocations are core allocations. The reason why part (iii) of (A.2) is established here on the aggregate production set in Theorem 2 is because the distributivity property is not preserved by summation.

4.- CLASSICAL EQUILIBRIA WITH POSITIVE PROFITS

Let us address now the question of whether there exist Classical Equilibria with (strictly) positive profit rates, in this static framework. The answer depends very much on the specifics of the model under consideration. The existence of Classical Equilibria with positive profits cannot be ensured in general, although it might be so under certain circumstances. We shall briefly discuss here a particular case of the model in Section 2, where positive profits and Classical Equilibria turn out to be compatible.

Note first that in a Classical Equilibrium with positive profits, no consumption of produced commodities with positive prices can occur. This is so because in this case consumers devote all their initial endowments to production activities (in order to maximize wealth), and hence in equilibrium these commodities are actually used up by firms. Unless this condition holds, zero would be the only profit rate compatible with the existence of equilibrium. Hence it is natural to consider again the case where initial endowments consist only of production factors which do not enter the preferences of consumers.

The following result is obtained:

Theorem 3.- Under assumptions (A.1) to (A.4), for any given $\rho \in \mathcal{R}$, there exists a Classical Equilibrium relative to ρ.

Theorem 3 says that there exists an equilibrium with a rate of profits $r(\mathbf{p}^*, \mathcal{Y}^*) = \rho$, for any pre-established $\rho \in \mathcal{R}$. This points out a strong indeterminacy in the model: there may be many possible equilibria with different profit rates and, consequently, different income distributions, employment levels, etc. Profits are to be interpreted as the rents of those scarce factors which are required in order to carry out production activities. The profit rate may be seen as a parameter of the way in which the total surplus is distributed across agents. Yet the informative content of this parameter is rather ambiguous, except in very specific models [as in Sraffa (1960)].

The following Corollary is of interest:

Corollary 2.- Let an economy satisfying assumptions (A.1), (i) and (ii) of (A.2), (A.3) and (A.4). Then,

(i) If N_1 is empty (i.e., all firms exhibit constant returns to scale), for every $\rho \in \mathcal{R}$, there is a Pareto Optimal Classical Equilibrium relative to ρ.

(ii) If moreover $Y_0 = \sum_{j=1}^{n} Y_j$ is distributive, then there exists a Classical Equilibrium relative to ρ which is Pareto optimal, for every $\rho \in \mathcal{R}$.

Corollary 3 is simply a rephrasing of Corollary 1 and Theorem 2 within this new context (so the proof will be omitted). Nevertheless, it is worth noticing that it shows the existence of efficient equilibrium allocations in the presence of positive profits.

5.- FINAL REMARKS

We have presented a model of a market economy with competitive features, where there may be several industries with non-decreasing returns to scale. Rational consumers set up firms by applying their resources to the most profitable production activities, and maximize utility subject to their budget sets. Firms' behaviour consists of maximizing profits at given prices. A Classical Equilibrium is a situation where all these actions are simultaneously feasible. Existence results have been provided, distinguishing between the cases of zero and positive equilibrium profits. Some efficiency properties have also been analyzed.

It is interesting to note the connection between this model and some other general equilibrium models. When production sets are convex cones (constant returns to scale), a Canonical Classical Equilibrium corresponds to a standard Arrow-Debreu competitive equilibrium. When the economy consists of a single distributive firm, a Canonical Classical Equilibrium is a Social Equilibrium à la Scarf. Many equilibrium models in the classical tradition (von Neumann, Leontief, Sraffa, and their variants) may be interpreted as particular cases of this one.

Let us conclude by commenting on the treatment given here to commodities, transactions and profits.

The rationale of the division between Produced Commodities and New Goods is twofold. On the one hand it improves the descriptive power of the model, stressing the picture of a society which develops production activities in order to obtain commodities which are not available. On the other, it provides a natural basis for the analysis of Classical

Equilibria with positive profits. These advantages, however, are bought at a cost: the abstract requirement introduced in assumption (A.3) (the "cheaper point" condition). If we dispense with this classification among commodities, and substitute (A.3) by the more standard assumption of "$\omega_i \in \text{int}X_i$ for all i", then the model is still valid, but then the only Classical Equilibria which may occur are the Canonical ones (unless one imposes some arbitrary restrictions on the trading process). It is worth stressing that such a division of commodities plays no role in the analysis of Canonical Classical Equilibria developed in section 3.

It was already pointed out (see Remark 1) that one may interpret the model as including a <u>sequence</u> of transactions within the period. Investment decisions are taken first, then firms are created and production occurs, and finally income is realized and consumption takes place. This sequential feature is relevant again for the analysis of equilibria with positive profits. For suppose that consumers can spend, before production takes place, the profits that will result from production activities. Then, whenever $r(\mathbf{p}, y) > 0$, consumers will use their profits $\mathbf{p}\omega r(\mathbf{p}, y)$ to buy additional endowments, and spend the additional future yields to buy even more endowments, and will repeat this process again and again before deciding their consumption. But this income cannot be realized as an equilibrium, so that the only possible equilibria would be the Canonical ones.

Needless to say that a sharper way of capturing this sequential character of transactions would be to set up a two-period model. Yet, this framework would necessarily involve some equilibrium "interest rate", which would obscure the nature of pure profits as rents of scarce factors, that we wanted to stress.

Appendix: Proof of the Theorems

Let $\mathbb{P}$ denote the price simplex in $\mathbb{R}^{\ell}$, that is,

$$\mathbb{P} \equiv \{ \mathbf{p} \in \mathbb{R}^{\ell}_{+} \; / \; \sum_{i=1}^{\ell} p_i = 1 \}$$

A <u>Pricing</u> <u>Rule</u> for the *jth* firm is a (set-valued) mapping ϕ_j applying the set of efficient production plans into $\mathbb{P}$. For any efficient production plan $\mathbf{y}_j \in \mathfrak{F}_j$, $\phi_j(\mathbf{y}_j)$ should be interpreted as the set of price vectors found "acceptable" by the *jth* firm when producing $\mathbf{y}_j$. In other words, the *jth* firm is in equilibrium whenever the pair $(\mathbf{y}_j, \mathbf{p})$ is in the graph of ϕ_j. Observe that under assumption (A.2) the set of weakly efficient production plans consists exactly of those points in the boundary of Y_j.

A mapping $\phi_j : \mathfrak{F}_j \longrightarrow \mathbb{P}$ is a <u>Loss-Free</u> <u>Pricing</u> <u>Rule</u> if $\mathbf{q}\mathbf{y}_j \geq 0$, for every $\mathbf{q} \in \phi_j(\mathbf{y}_j)$.

The following result is well established in the literature [see for instance Bonnisseau & Cornet (1988, Th. 2.1')]:

Lemma 1.- Let E stand for an economy satisfying assumptions (A.1), (i) and (ii) of (A.2), and (A.3), and suppose that: (1) ϕ_j is a loss-free pricing rule, upper hemicontinuous with nonempty compact and convex values, for all *j*; and (2) W_i is a continuous function for all *i*. Then, there exist a price vector $\mathbf{p}^*$ and an allocation $[(\mathbf{x}_i^*), \mathcal{y}^*]$ such that:

(a) $\mathbf{x}_i^* \in \xi_i(\mathbf{p}^*, y^*)$, $\forall\ i$

(b) $\mathbf{p}^* \in \bigcap_{j=1}^{n} \phi_j(\mathbf{y}_j^*)$

(c) $\sum_{i=1}^{m} \mathbf{x}^* - \omega = \sum_{j=1}^{n} \mathbf{y}_j^*$.

In order to prove the Theorem, let us define two particular pricing rules, which do not depend on convexity: Average Cost Pricing and Constrained Profit Maximization.

Average Cost Pricing is defined by:

a) If $\mathbf{y}_j \neq 0$,

$$\psi_j^{AC}(\mathbf{y}_j) \equiv \{\ \mathbf{q} \in \mathbb{P}\ /\ \mathbf{q}\ \mathbf{y}_j = 0\ \}$$

b) If $\mathbf{y}_j = 0$,

$$\psi_j^{AC}(0) \equiv clco\ \{\ \mathbf{q} \in \mathbb{P}\ /\ \exists\ \{\mathbf{q}^\nu, \mathbf{y}_j^\nu\} \subset \mathbb{P} \times [\mathfrak{F}_j \setminus \{0\}] \text{ with } \{\mathbf{q}^\nu, \mathbf{y}_j^\nu\} \longrightarrow (\mathbf{q}, 0) \text{ and } \mathbf{q}^\nu \mathbf{y}_j^\nu = 0\ \}$$

(where $clco$ {.} denotes the closed convex hull of {.}).

Constrained Profit Maximization is given by:

$$\psi_j^{CPM}(\mathbf{y}_j) \equiv \{\ \mathbf{q} \in \mathbb{P}\ /\ \mathbf{q}\mathbf{y}_j \geq \mathbf{q}\mathbf{y}'_j,\ \ \forall\ \mathbf{y}'_j \in Y_j \text{ with } \mathbf{a}'_j \geq \mathbf{a}_j\ \}$$

Thus, ψ_j^{CPM} pictures the *jth* firm as selecting, for each given efficient production plan $\mathbf{y}_j$, prices such that it is not possible to obtain higher profits within the set of production plans which make use of equal or fewer inputs.

The following lemmata will lead up to the main result:

Lemma 2.- Scarf (1986, Th. 1)

Under assumption (A.2), for each point $(b_j, a_j) \in \mathfrak{F}_j$ there exists $q \in \mathbb{P}$ such that

$$0 = q(b_j, a_j) \geq q(b'_j, a'_j)$$
$$\forall\ (b'_j, a'_j) \in Y_j \text{ such that } a'_j \geq a_j.$$

Lemma 3.- Under assumption (A.2), ψ_j^{CPM} is a closed correspondence, with nonempty convex values.

<u>Proof.-</u>

Under assumption (A.2), $\psi_j^{CPM}(y_j)$ is clearly convex, for each $y_j \in \mathfrak{F}_j$. Lemma 2 ensures that it is also non-empty.

To see that the graph is closed[5], let $[(b^o, a^o), p^o]$ be an arbitrary point in $\mathfrak{F}_j \times \mathbb{P}$, and let $\{ (b^\nu, a^\nu), p^\nu \}$ be a sequence converging to $[(b^o, a^o), p^o]$, such that $[(b^\nu, a^\nu), p^\nu] \in \mathfrak{F}_j \times \mathbb{P}$, and p^ν belongs to $\psi_j(b^\nu, a^\nu)$, for all ν. Suppose, by way of contradiction, that p^o is not in $\psi_j(b^o, a^o)$. Then there exists $(b', a') \in Y_j$, with $a' \geq a^o$,

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[5] This part goes along the lines of Lemma 1 in Dehez & Drèze (1988 a).

such that $p^o(b', a') > p^o(b^o, a^o)$. This implies that, for ν big enough ($\nu > \nu'$, say), we also have:

$$p^\nu(b', a') > p^\nu(b^\nu, a^\nu)$$

If $a' \geq a^\nu$ and $\nu > \nu'$, this contradicts the assumption that p^ν belongs to $\psi_j(b^\nu, a^\nu)$. Suppose that this is not the case. We have now two possibilities. First, $a^o = 0$, and consequently $a' = 0$. Then it follows that $b^o = 0 \geq b'$, and hence the inequality above cannot hold. Suppose then that $a^o < 0$, and construct a new point (b'', a'') in Y_j as follows:

(i) $a''_t = a'_t + \varepsilon$, if $a'_t < 0$ (where $\varepsilon > 0$ is a scalar arbitrarily small), and $a''_t = 0$, otherwise; and

(ii) $b''_t = b'_t - \delta_t$ (where $\delta_t \geq 0$ is a scalar arbitrarily small).

Since Y_j is a closed and comprehensive set, these scalars can always be chosen so that (b'', a'') lies in Y_j, and $p^o(b'', a'') > p^o(b^o, a^o)$. Note that, by construction, $a'' > a' \geq a^o$.

Now observe that for ν big enough, there will be points (b^ν, a^ν) close to (b^o, a^o) such that $a'' \geq a^\nu$. For these points we have:

$$p^\nu(b'', a'') > p^\nu(b^\nu, a^\nu)$$

while $p^\nu \in \psi_j(b^\nu, a^\nu)$, contradicting the hypothesis.

■

Theorem 1.- Let E be an economy satisfying assumptions (A.1) to (A.3).

Then:

(i) A Canonical Classical Equilibrium $[p^*, (x_i^*), y^*]$ exists.

(ii) There is no feasible allocation $[(x_i'), y']$, such that $u_i(x_i') \geq u_i(x_i^*)$, $\forall$ *i*, with at least a strict inequality, and $a_j' \geq a_j^*$ for all *j*.

Proof.-

(i) Let $\hat{E}$ be an economy whose data (consumers, technology and initial endowments) are identical to those in E, but in which consumers and firms behave according to the following pattern:

a) The *ith* consumer maximizes utility at given prices, within her budget set, which is defined by the following wealth function:

$$\hat{W}_i(p, y) = p\omega_i$$

b) The *jth* firm's feasible set corresponds to Y_j and behaves according to a pricing rule ϕ_j, which is defined as follows:

$$\phi_j(y_j) \equiv \psi_j^{AC}(y_j) \cap \psi_j^{CPM}(y_j)$$

It is well known that, under assumption (A.2), ψ_j^{AC} is an upper hemicontinuous correspondence with nonempty, compact and conve values. From Lemma 3 it follows that ψ_j^{CPM} is also an upper hemicontinuous correspondence (because it is a compact-valued correspondence with closed graph), with convex and compact values. Therefore, ϕ_j is an upper hemicontinuous, convex-valued and loss-free pricing rule. Lemma 2 ensures that ϕ_j is nonempty valued.

Since $\hat{E}$ also satisfies assumptions (A.1), (A.2) and (A.3), and $\hat{W}_i$ is continuous for all i, we can apply Lemma 1 which ensures the existence of a price vector $\mathbf{p}^*$ and an allocation $[(\mathbf{x}_i^*), \mathcal{y}^*]$ such that:

(a) $\mathbf{x}_i^* \in \xi_i(\mathbf{p}^*, \mathcal{y}^*)$, for all i;

(b) $\mathbf{p}^* \in \bigcap_{j=1}^{n} \phi_j(\mathbf{y}_j^*)$; and

(c) $\sum_{i=1}^{m} \mathbf{x}^* - \omega = \sum_{j=1}^{n} \mathbf{y}_j^*$.

It is immediate to check that this corresponds to a Canonical Classical Equilibrium for the original economy.

(ii) The argument here is standard, and included for the sake of completeness:

Let $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathcal{y}^*]$ be a Canonical Classical Equilibrium, and suppose now that there is another feasible allocation $[(\mathbf{x}_i'), \mathcal{y}']$ such that: (1) $u_i(\mathbf{x}_i') \geq u_i(\mathbf{x}_i^*)$ for every i, with a strict inequality for some consumer; and (2) $\mathbf{a}_j' \leq \mathbf{a}_j^*$ for every j. Since this allocation is feasible, it must be the case that

$$\sum_{i=1}^{m} \mathbf{x}_i' \leq \omega + \sum_{j=1}^{n} \mathbf{y}_j'$$

Now notice that non-satiation implies that

$$\mathbf{p}^* \sum_{i=1}^{m} \mathbf{x}_i' > \mathbf{p}^* \sum_{i=1}^{m} \mathbf{x}_i^* = \mathbf{p}^* \omega + \mathbf{p}^* \sum_{j=1}^{n} \mathbf{y}_j^*$$

Therefore substituting we get

$$p^* \sum_{j=1}^{n} y'_j > p^* \sum_{j=1}^{n} y^*_j = 0$$

This implies that there is some j for which $p^* y'_j > 0$. This j cannot exist if we require that $a'_j \geq a^*_j$, according to the definition of Canonical Classical Equilibrium. Therefore, such an allocation cannot exist.

■

Theorem 2.- Let E be an economy satisfying assumptions (A.1), (i) and (ii) of (A.2), (A.3) and (A.4). Suppose furthermore that $Y_0 = \sum_{j=1}^{n} Y_j$ is distributive. Then there exists a Canonical Classical Equilibrium which is in the core.

Proof.-

The proof will be divided into two steps.

(i) Consider an economy E_0 identical to E in all respects except in that we substitute all individual firms by a single aggregate one Y_0. Clearly, Y_0 satisfies (A.2).

Now, for each $y_0 \in \partial Y_0$, define: $\phi_0(y_0) \equiv \psi_0(y_0) \cap \mathbb{P}$ (with an obvious meaning). In view of Lemma 2, this mapping is an upper hemicontinuous correspondence, with nonempty, convex and compact values. Hence, Theorem 1 ensures the existence of a Canonical Classical Equilibrium for the E_0 economy.

Let us show now that this equilibrium for E_0 actually corresponds to a Canonical Classical Equilibrium for the original economy. First notice that, by construction, $\mathbf{y}_0^*$ can be expressed as $\sum_{j=1}^{n} \mathbf{y}_j^*$ with $\mathbf{y}_j^* \in Y_j$ for all j. It follows that if $\mathbf{y}_0^*$ maximizes profits at prices $\mathbf{p}^*$ within the set $\{ \mathbf{y}_0 \in Y_0 \ / \ \mathbf{a}_0 \geq \mathbf{a}_0^* \}$, it must be the case that <u>every</u> $\mathbf{y}_j^*$ maximizes profits at $\mathbf{p}^*$ in the set $\{ \mathbf{y}_j \in Y_j \ / \ \mathbf{a}_j \geq \mathbf{a}_j^* \}$, $j = 1, 2, \ldots, n$. For suppose not, that is, suppose that there exists $\mathbf{y}'_k$ with $\mathbf{a}'_k \geq \mathbf{a}_k^*$ such that $\mathbf{p}^* \mathbf{y}'_k > \mathbf{p}^* \mathbf{y}_k^*$; then, substituting $\mathbf{y}_k^*$ by $\mathbf{y}'_k$ in $\mathbf{y}_0^*$ we would get:

$$\mathbf{p}^* \left[\sum_{j \neq k} \mathbf{y}_j^* + \mathbf{y}'_k \right] > \mathbf{p}^* \mathbf{y}_0^*$$

with $\left[\sum_{j \neq k} \mathbf{y}_j^* + \mathbf{y}'_k \right] \in \{ \mathbf{y}_0 \in Y_0 \ / \ \mathbf{a}_0 \geq \mathbf{a}_0^* \}$, contradicting the hypothesis.

Now observe that, since $\mathbf{p}^*\mathbf{y}_0^* = 0$, and $0 \in Y_j$ for all j, it follows that $\mathbf{p}^*\mathbf{y}_j^* = 0$ for all j. Then, the aggregation of individual firms into a single one does not affect consumers' wealth functions. This implies that the allocation $[(\mathbf{x}_i^*), \mathbf{y}_0^*]$ can be dis-aggregated into an allocation of the original economy $[(\mathbf{x}_i^*), \mathcal{Y}^*]$ such that $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathcal{Y}^*]$ is a Canonical Classical Equilibrium.

(ii) Let $[(\mathbf{x}_i^*), \mathcal{Y}^*]$ be the allocation constructed in part (i), and suppose that there is an allocation $[(\mathbf{x}'_i), \mathcal{Y}']$ and a coalition S of consumers such that: (a) $u_i(\mathbf{x}'_i) \geq u_i(\mathbf{x}_i^*)$ for every $i \in S$, with a strict inequality for some consumer; and (b) $\sum_{i \in S} \mathbf{x}'_i - \sum_{i \in S} \omega_i = \sum_{j=1}^{n} \mathbf{y}'_j$. Now notice that non-satiation implies that

$$p^* \sum_{i \in S} x'_i \; > p^* \sum_{i \in S} x^*_i \; = p^* \sum_{i \in S} \omega_i$$

Therefore substituting we get $\sum_{j=1}^{n} p^* y'_j \; > 0$.

But this is not possible, since: (a) Feasibility and assumption (A.4) imply that $\sum_{j=1}^{n} a'_j \geq - \sigma$ (the aggregate endowment of Produced Commodities); and (b) $\sum_{j=1}^{n} y^*_j$ is a profit maximizing combination of production plans, subject to the restriction $\sum_{j=1}^{n} a'_j \; \geq \; \sum_{j=1}^{n} a^*_j \; = - \sigma$, with $p^* \sum_{j=1}^{n} y^*_j \; = 0$ [in view of (A.4), the definition of Canonical Classical Equilibrium, and the way in which $\mathcal{y}^*$ has been chosen].

The proof is in this way completed.

■

Theorem 3.- Under assumptions (A.1) to (A.4), for any given $\rho \in \mathcal{R}$ there exists a Classical Equilibrium relative to ρ.

Proof.-

(i) Consider an economy $E(\rho)$, for $\rho > 0$, which is identical to the original one except in the following:

a) The *ith* consumer's initial endowments are given by:

$$\omega_i(\rho) \;\; \equiv \;\; \omega_i(1+\rho)$$

b) The *jth* production set, is now defined as:

$$Y_j(\rho) \equiv \{ \mathbf{s} \in \mathbb{R}^{\ell} \;/\; \mathbf{s} = [\mathbf{b}_j, \mathbf{a}_j(1+\rho)], \quad \text{with } (\mathbf{b}_j, \mathbf{a}_j) \in Y_j \}$$

whose elements will be denoted by $\mathbf{y}_j(\rho)$.

This economy satisfies all assumptions (A.1) to (A.3), so that we can apply Theorem 1 which ensures the existence of a Canonical Classical Equilibrium for the E(ρ) economy, $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathcal{Y}^*(\rho)]$.

(ii) Let us now show that $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathcal{Y}^*]$ is a Classical Equilibrium relative to ρ for the original economy. First note that $[(\mathbf{x}_i^*), \mathcal{Y}^*]$ is an attainable allocation. To see this observe that, by assumption,

$$\sum_{i=1}^{m} \mathbf{x}_i^* - \omega(1+\rho) - \sum_{j=1}^{n} \mathbf{y}_j^*(\rho) = \mathbf{0} \qquad [1]$$

The structure of the model and assumption (A.4) allow us to write

$$\sum_{i=1}^{m} \mathbf{x}_i^* = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}, \quad \omega(1+\rho) = \begin{bmatrix} \mathbf{0} \\ \sigma(1+\rho) \end{bmatrix}, \quad \sum_{j=1}^{n} \mathbf{y}_j^*(\rho) = \begin{bmatrix} \mathbf{b} \\ \mathbf{a}(1+\rho) \end{bmatrix}$$

Then equation [1] can be rewritten as follows:

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \sigma(1+\rho) \end{bmatrix} - \begin{bmatrix} \mathbf{b} \\ \mathbf{a}(1+\rho) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which implies that $\mathbf{c} = \mathbf{b}$, $-\sigma(1+\rho) = \mathbf{a}(1+\rho)$, and, consequently, $-\sigma = \mathbf{a}$. Therefore it follows that

$$\sum_{i=1}^{m} \mathbf{x}_i^* - \omega - \sum_{j=1}^{n} \mathbf{y}_j^* = 0$$

It is easy to see that, for each *j*, $\mathbf{p}^*\mathbf{y}_j^* \geq \mathbf{p}^*\mathbf{y}_j$ for all $\mathbf{y}_j$ such that $\mathbf{a}_j \geq \mathbf{a}_j^*$. For suppose not, that is, suppose that there exists a firm *j* and a production plan $\mathbf{y}'_j$ such that $\mathbf{p}^*\mathbf{y}'_j > \mathbf{p}^*\mathbf{y}_j^*$, with $\mathbf{a}'_j \geq \mathbf{a}_j^*$. In that case we would also have that $\mathbf{p}^*\mathbf{y}'_j(\rho) > \mathbf{p}^*\mathbf{y}_j^*(\rho)$, against the hypothesis that $[\mathbf{p}^*, (\mathbf{x}_i^*), \mathscr{y}^*(\rho)]$ is a Classical Equilibrium for the E(ρ) economy.

It remains to show that $\mathbf{x}_i^*$ is the *ith* consumer's demand in the original economy. But this follows immediately from the way the E(ρ) economy has been constructed.

$\blacksquare$

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