EXISTENCE AND EFFICIENCY OF EQUILIBRIUM IN ECONOMIES WITH INCREASING RETURNS TO SCALE: AN EXPOSITION*

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A B S T R A C T

The purpose of this paper is to offer an exposition of the results on the existence and optimality of equilibria when production sets are not assumed to be convex, in a general equilibrium framework. We aim at providing a formal and systematic account of the main results available, rather than survey the literature. Besides presenting an abstract model, where firms' behaviour is described by general pricing rules, we analyze the family of Loss-free Pricing Rules (focusing on Profit Maximization, both constrained and unconstrained, and Average Cost Pricing), and the Marginal Pricing Rule and other regulation policies. Then, we discuss the efficiency problem, referring to both the first and second welfare theorems.

KEYWORDS: Increasing Returns, Equilibrium, Optimality.

1.- INTRODUCTION.

The standard Arrow-Debreu-MacKenzie general equilibrium model of a competitive economy, provides a basic tool for the understanding of the functioning of competitive markets. It allows us to give a positive answer to the old question concerning the capability of prices and markets to coordinate the economic activity in a decentralized framework. This model shows that, under a set of well specified assumptions, markets are in themselves sufficient institutions for the efficient allocation of resources. This may be called the Invisible-Hand Theorem, and summarizes the most relevant features of competitive markets: the equilibria constitute a nonempty subset of the set of efficient allocations.

The existence of a competitive equilibrium is usually obtained by applying a fixpoint argument. The strategy of the proof consists of identifying the set of competitive equilibria with the set of fixpoints of a suitable mapping, and making use of Kakutani's Fixpoint Theorem. For this approach to work, one has to be able to ensure that the set of attainable allocations of the economy is nonempty and bounded, and that the excess demand mapping is an upper hemicontinuous correspondence, with nonempty, closed and convex values. The convexity of preferences and of consumption and production sets allows one to obtain an excess demand mapping with such properties, when agents behave as payoff maximizers at given prices.

On the other hand, the efficiency of competitive equilibria is derived from two basic features. The first one refers again to the fact that agents behave as payoff maximizers at given prices, so that each agent equates her marginal rates of transformation to the relative prices (and hence in equilibrium they become equal for *all agents* and *all commodities*). The second one is that *each variable affecting the payoff function of an individual has associated a price, and belongs to her choice set* (so that prices turn out to be sufficient information, enabling the exploitation of all benefits derived from production and exchange). The equalization of prices and marginal rates of transformation is a necessary condition for optimality, which under the assumption of convex preferences and choice sets (and complete markets) turns out to be sufficient as well.

Price-taking behaviour, perfect information, complete markets and quasi-concave payoff functions defined over convex choice sets are thus the key elements for the Invisible-Hand Theorem to hold. This in turn points out that there are many relevant instances in which this Theorem does not work, either because competitive equilibria do not belong to the set of efficient allocations, or because they simply do not exist (externalities, asymmetric information, oligopolistic competition, etc.). The presence of increasing returns to scale (or more general forms of non-convex technologies) is a case in point.

The convexity of production sets can be derived from the combination of two primitive hypotheses: <u>Additivity</u> and <u>Divisibility</u>. The additivity assumption says that if two production plans are technologically feasible, a new production plan consisting of the sum of these two will also be possible. Divisibility says that if a production plan is feasible, then any production plan consisting of a reduction in the scale of the former will also be feasible (non-decreasing returns to scale). When these hypotheses hold, production sets turn out to be convex cones. While the additivity assumption seems hard to reject on economic or engineering grounds[1], the divisibility assumption is much more debatable, both theoretical and empirically. Hence the main sources of nonconvexities in production can be related to a failure in the divisibility assumption, that is, to the presence of indivisibilities, fixed costs or increasing returns to scale [see Mas-Colell (1987 IV-VI) and Guesnerie (1990, 5.1) for a brief discussion concerning the origin and classes of nonconvexities].

General equilibrium models face serious difficulties in the presence of non-convex technologies, when there are finitely many firms and non-convexities are not negligible. Such difficulties are both analytical and theoretical and have mainly to do with the fact that the supply correspondence may not be convex-valued or even defined, so that the existence of a Nash equilibrium will typically fail. Hence, alternative techniques of analysis and different equilibrium concepts must be applied. In particular, profit maximizing behaviour at given prices and increasing returns turn out to be incompatible with the presence of active firms (since, in this case, the supply mapping will not be defined for non-zero

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(1) Even though the theory allows for general convex sets, it is difficult to explain the lack of additivity. In some cases it is attributed to the existence of some limitation of inputs. But this cannot be part of the technological description of the economy, once all commodities are taken into account. Furthermore, allowing for some input restrictions in the description of production sets implies that we are admitting the existence of a procedure of allocating such scarce inputs, outside the market mechanism; in this case the first welfare theorem cannot be applied (it would be possible that a different allocation of these scarce inputs would result in a Pareto superior state).

outputs). This implies, that if we want to analyze a general equilibrium model allowing for non-convex technologies, we must permit the firms to follow more general rules of behaviour, and suitably re-define the equilibrium notion. This will, however, imply that the identification between equilibrium and optimum will no longer hold (the Invisible-Hand Theorem now becomes split into two different halves). Thus the existence of equilibria under nonconvex technologies, and the analysis of their optimality properties become now two separate and substantive questions.

The modern approach to these problems consists of building up a general equilibrium model which constitutes a genuine extension of the standard one. For that, an equilibrium for the economy is understood as a price vector, a list of consumption allocations and a list of production plans such that: (a) the consumers maximize their preferences subject to their budget constraints; (b) each individual firm is in "equilibrium" at those prices and production plans; and (c) the markets for all goods clear. It is the nature of the equilibrium condition (b) which establishes the difference with respect to the Walrasian model. The central question becomes now the following: How to model the behaviour of non-convex firms (according to relevant positive and/or normative criteria), in such a way that an equilibrium existence theorem applies.

A very general and powerful way of dealing with this question consists of associating the equilibrium of firms with the notion of a Pricing Rule, rather than to that of a supply correspondence. A Pricing Rule is a mapping applying each firm's set of efficient production plans on the price space. The graph of such a mapping describes the prices-production pairs which a firm finds "acceptable" (a pricing rule may be thought of as the inverse mapping of a generalized "supply correspondence"). The advantage of formulating the problem in this way is twofold: (1) The notion of a Pricing rule is an abstract construct which allows to model different types of behaviour, and thus to analyze situations were profit maximization is not applicable. (2) These mappings may be upper hemicontinuous and convex-valued, even when the supply correspondence is not so, making it possible to use a fixpoint argument (on the "inverse supply" mapping), in order to get the existence of an equilibrium.

As for the ways of modelling the behaviour of non-convex firms in terms of pricing rules, let us point out that both positive and normative approaches are possible. Positive models intend to describe plausible behaviours of these firms in the context of unregulated markets, while Normative Models typically associate non-convex firms with public utilities (which may be privately owned but regulated). Models within the first category include Constrained Profit Maximization (i.e., situations where firms maximize profits in the presence of some type of quantity constraint), and Average Cost (or more generally, Mark-up) Pricing. Normative models concentrate over two main pricing rules: Marginal (cost) Pricing, and Regulation under Break-Even Constraints (including the case of two-part tariffs, which may satisfy both criteria). These pricing rules constitute attempts at getting First and Second Best Efficient equilibria. We elaborate on all this later on.

The purpose of this paper is to offer an exposition of the results on the existence and optimality of equilibria when production sets are not assumed to be convex, in a general equilibrium framework. We aim at providing a formal and systematic account of the main results available, rather than to survey the literature. There is a number of papers which survey the recent literature on this area. Among them let us mention the following: Mas-Colell (1987) contains a simplified exposition of the problems and lines of research related to equilibrium models with increasing returns. Cornet (1988) provides a short review to general equilibrium with non-convex technologies, following the Pricing Rule approach; his paper is an Introduction to the special issue of the Journal of Mathematical Economics where many of the recent contributions appear. Dehez (1988) and Brown (1991) are much more comprehensive papers, well articulated and informative. Dehez's paper focuses more on interpretive issues, while Brown's work contains a very good systematization of the analytical underpinnings of these models. Guesnerie (1990) and Quinzii (1992) provide illuminating discussions of the normative aspects of the topic.

The paper is organized as follows. Section 2 contains the base-model and states an existence result for general pricing rules. Sections 3 and 4 are devoted to presenting a series of equilibrium models which are particular cases of the one developed in Section 2. These specific models illustrate the flexibility of the pricing rule approach for the analysis of general equilibrium, and add some flesh to that abstract framework.

Section 3 refers to the family of Loss-free Pricing Rules (those in which the equilibrium of firms involves nonnegative profits), focusing on two main categories: Profit Maximization (constrained or unconstrained), and Average Cost Pricing. We shall emphasize here the positive approach. Then Section 4 concentrates on the Marginal Pricing Rule (a pricing rule satisfying the necessary conditions for optimality) and Two-Part Marginal Pricing and other Regulation Policies (Boiteaux-Ramsey prices and Aumann-Shapley values). Finally, Section 5 discusses the efficiency problem in economies with nonconvex production sets. We conclude each of these Sections with a paragraph containing "References to the Literature".

2.- A GENERAL EQUILIBRIUM MODEL WITH NONCONVEX TECHNOLOGIES.

We present in this Section a general equilibrium model where the convexity assumption on production sets has been dropped, and each firm's behaviour is modelled in terms of an abstract pricing rule[2]. Notation and concepts follow Debreu (1959), unless otherwise specified.

The abstract Pricing Rule approach has to cope with a number of problems when we come to analyze the existence of equilibrium. These problems, which are ones of technique and of substance, do not exist in the standard competitive world, and turn out to be interdependent and to appear simultaneously. Let us briefly comment on them, in order to clarify the nature of the assumptions we shall meet later on:

1) In the absence of convexity, the set of attainable allocations may not be bounded. This implies that some hypothesis on the compactness of this set must be introduced, if we want to be able to apply a fixpoint argument.

2) When firms do not behave as profit maximizers at given prices, they may suffer losses in equilibrium; this is the case of Marginal Pricing, which yields negative profits under increasing returns to scale. Hence some restriction on the distribution of wealth must be imposed in order to avoid difficulties for the survival of consumers (and the upper hemicontinuity of the demand mapping). Indeed the survival assumption

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(2) We follow closely the work in Villar (1994).

turns out to be a key element in the shaping of models with increasing returns.

3) If firms do not follow (unconstrained) Profit Maximization, equilibrium allocations will not be efficient in general. Even if there exists an equilibrium where non-convex firms follow Marginal Pricing, it may not be Pareto optimal (since the equalization between marginal rates of transformation and prices is not sufficient in this case).

4) Pricing rules cannot be totally arbitrary. In particular, each firm's pricing rule must exhibit some sensitivity with respect to changes in production, since an equilibrium price vector must belong to the intersection of all firms' pricing rules (think of the case of two firms, each of which only accepts a single price vector for any possible production plan, and in which both price vectors differ).

The convention for vector inequalities is: $\geq$, $>$, $\gg$.

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Consider an economy with ℓ perfectly divisible commodities, m consumers (indexed $i = 1, 2, \ldots, m$) and n firms (indexed $j = 1, 2, \ldots, n$). A point $\omega \in \mathbb{R}^{\ell}$ denotes the aggregate vector of initial endowments. The *jth* firm's production set is represented by a subset Y_j of $\mathbb{R}^{\ell}$, while $\mathfrak{F}_j$ denotes the *jth* firm's set of weakly efficient production plans, that is,

$$\mathfrak{F}_j \equiv \{ y_j \in Y_j \;/\; y'_j \gg y_j \implies y'_j \notin Y_j \}$$

$\mathfrak{F}$ will stand for the cartesian product of the n sets of weakly efficient production plans, that is, $\mathfrak{F} \equiv \prod_{j=1}^{n} \mathfrak{F}_j$. We shall denote by $\mathbb{P} \subset \mathbb{R}^{\ell}_{+}$ the standard price simplex, that is, $\mathbb{P} = \{ \mathbf{p} \in \mathbb{R}^{\ell}_{+} \;/\; \sum_{t=1}^{\ell} p_t = 1 \}$. For a point $\mathbf{y}_j \in \mathfrak{F}_j$ and a price vector $\mathbf{p} \in \mathbb{P}$, $\mathbf{py}_j$ gives us the associated profits.

Each firms' behaviour will now be defined in terms of a <u>Pricing</u> <u>Rule</u>. A Pricing Rule for the *jth* firm is usually defined as a mapping Φ_j applying the set of efficient production plans, $\mathfrak{F}_j$ into $\mathbb{R}^{\ell}_{+}$. For a point $\mathbf{y}_j \in \mathfrak{F}_j$, $\Phi_j(\mathbf{y}_j)$ has to be interpreted as the set of price vectors found "acceptable" by the *jth* firm when producing $\mathbf{y}_j$. In other words, the *jth* firm is in equilibrium at the pair $(\mathbf{y}_j, \mathbf{p})$, if $\mathbf{p} \in \Phi_j(\mathbf{y}_j)$. Even though in most of the cases the *jth* firm's Pricing Rule only depends on y_j, we shall adopt the more general notion of firms' behaviour, by allowing for each firm's Pricing Rule to depend on other firms' actions and "market prices". To do this, let $\bar{\mathbf{y}} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ denote a point in $\mathfrak{F}$. Then,

<u>Definition</u> <u>2.1.-</u> A **Pricing Rule for the jth firm** is a correspondence,

$$\phi_j : \mathbb{P} \times \mathfrak{F} \longrightarrow \mathbb{P}$$

which establishes the *jth* firm set of admissible prices, as a function of "market conditions".

That is, $\mathbf{y}_j$ is an equilibrium production plan for the *jth* firm at prices $\mathbf{p}$, if and only if, $\mathbf{p} \in \phi_j(\mathbf{p}, \bar{\mathbf{y}})$ (where $\mathbf{y}_j$ is precisely the *jth* firm's production plan in $\bar{\mathbf{y}}$). As for interpretive purposes, we may think of a market mechanism in which there is an auctioneer who calls out both a price vector (to be seen as proposed market prices), and a vector of efficient production plans. Then, the *jth* firm checks out whether the pair $(\mathbf{p}, \mathbf{y}_j)$ agrees with her objectives (formally, $[(\mathbf{p}, \bar{\mathbf{y}}), \mathbf{p}]$ belongs to the graph of ϕ_j).

A situation in which all firms find acceptable the proposed combination between market prices and production plans is called a <u>production equilibrium</u>. Formally:

<u>Definition 2.2.-</u> We shall say that a pair $(\mathbf{p}, \bar{\mathbf{y}}) \in \mathbb{P} \times \mathfrak{F}$ is a **Production Equilibrium** if $\mathbf{p} \in \bigcap_{j=1}^{n} \phi_j(\mathbf{p}, \bar{\mathbf{y}})$.

Observe that different firms may follow different pricing rules. Furthermore, the pricing rule "may be either endogenous or exogenous to the model, and that it allows both price-taking and price-setting behaviors" [Cf. Cornet (1988, p. 106)]. We analyze alternative pricing rules in the next Sections.

The *ith* consumer is characterized by a tuple, $[\, X_i, u_i, r_i, \omega_i \,]$, where X_i, u_i, ω_i stand for the *ith* consumer's consumption set, utility function and initial endowments, respectively; by definition, $\sum_{i=1}^{m} \omega_i = \omega$. r_i denotes the *ith* consumer's wealth. To be precise, r_i is a mapping applying $\mathbb{P} \times \mathbb{R}^{\ell n}$ into $\mathbb{R}$ so that, for each pair $(\mathbf{p}, \bar{\mathbf{y}})$, $r_i(\mathbf{p}, \bar{\mathbf{y}})$ gives us the *ith* consumers' wealth.

Given a price vector $\mathbf{p}$, and a vector of production plans $\bar{\mathbf{y}} \in \mathfrak{F}$, the *ith* consumer's behaviour is obtained by solving the following program:

$$\begin{aligned} &\text{Max. } u_i \\ &\text{s.t.:} \\ &\quad \mathbf{x}_i \in X_i \\ &\quad \mathbf{p}\,\mathbf{x}_i \leq r_i(\mathbf{p}, \bar{\mathbf{y}}) \end{aligned}$$

Let $(\mathbf{p}, \bar{\mathbf{y}}) \in \mathbb{P} \times \mathfrak{Y}$ be given. Then, consumers' behaviour can be summarized by an aggregate net demand correspondence, that can be written as $\xi(\mathbf{p}, \bar{\mathbf{y}}) = d(\mathbf{p}, \bar{\mathbf{y}}) - \{\ \omega\ \}$, where $d(\mathbf{p}, \bar{\mathbf{y}}) \equiv \sum_{i=1}^{m} d_i(\mathbf{p}, \bar{\mathbf{y}})$, and d_i stands for the *ith* consumer's demand correspondence [i.e., $d_i(\mathbf{p}, \bar{\mathbf{y}})$ is the set of solutions to the program above for $(\mathbf{p}, \bar{\mathbf{y}})$].

Remark 2.1.- Observe that since consumers' choices depend on market prices and firms' production, we may think of each ϕ_j as also being dependent on consumers' decisions, that is,

$$\phi_j(\mathbf{p}, \bar{\mathbf{y}}) = \Theta_j[\mathbf{p}, \bar{\mathbf{y}}, \xi(\mathbf{p}, \bar{\mathbf{y}})].$$

This provides enough flexibility to deal with market situations in which firms' target payoffs may depend on demand conditions (as it is the case for Boiteaux-Ramsey prices).

The set of attainable allocations is given by:

$$\mathcal{A}(\omega) \equiv \{\ [(\mathbf{x}_i), \bar{\mathbf{y}}] \in \prod_{i=1}^{m} X_i \times \prod_{j=1}^{n} Y_j \ /\ \sum_{i=1}^{m} \mathbf{x}_i - \omega \leq \sum_{j=1}^{n} \mathbf{y}_j\ \}$$

The projection of $\mathcal{A}(\omega)$ on the space containing Y_j gives us the *jth* firm's set of attainable productions.

Consider now the following assumptions:

A.1.- For each firm $j = 1, 2, \ldots, n$, Y_j is a closed subset of $\mathbb{R}^{\ell}$, such that $0 \in Y_j$, and $Y_j - \mathbb{R}^{\ell}_{+} \subset Y_j$.

A.2.- For each given vector of initial endowments $\omega \in \mathbb{R}^{\ell}$, and every $(x_i) \in \prod_{i=1}^{m} X_i$ the *jth* firm's set of attainable productions is compact, $j = 1, 2, \ldots, n$.

A.3.- For each $i = 1, 2, \ldots, m$: (a) X_i is a closed and convex subset of $\mathbb{R}^{\ell}$, bounded from below; (b) $u_i : X_i \longrightarrow \mathbb{R}$ is a continuous and quasi-concave function, which satisfies Local Non-Satiation; and (c) $\omega_i \in X_i$ and there exists $x_i^{o} \in X_i$ such that $x_i^{o} \ll \omega_i$.

A.4.- $r_i : \mathbb{P} \times \mathbb{R}^{\ell n} \longrightarrow \mathbb{R}$ is a continuous function, quasi-concave in the second argument, and such that:

(a) For every $(\mathbf{p}, \bar{\mathbf{y}}) \in \mathbb{P} \times \mathbb{R}^{\ell n}$, we have:

$$\sum_{i=1}^{m} r_i(\mathbf{p}, \bar{\mathbf{y}}) = \mathbf{p}(\omega + \sum_{j=1}^{n} \mathbf{y}_j)$$

(b) For $\bar{\mathbf{y}} = 0$, $r_i(\mathbf{p}, 0) = \mathbf{p}\omega_i$, $i = 1, 2, \ldots, m$.

Assumption (A.1) provides us with a suitable generalization of the standard axioms on production sets. Besides closedness, it assumes the possibility of inaction and free-disposal. Observe that <u>under (A.1) the set of weakly efficient production plans, $\mathfrak{F}_j$, consists exactly of those points in the boundary of Y_j.</u>

Assumption (A.2) says that it is not possible (either for the *jth* firm or for the economy as a whole) to obtain an unlimited amount of production out of a finite amount of inputs.

Assumption (A.3) is standard and needs little comment. It contains all what is required in order to ensure that the demand correspondence is upper hemicontinuous, with nonempty, closed and convex values, in the context of a private ownership economy, where firms' profits are nonnegative.

Assumption (A.4) establishes that, for every $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathbb{R}^{\ell n}$, each consumer's wealth is a continuous mapping, quasi-concave in $\bar{\mathbf{y}}$, such that: (a) Total wealth equals the value of the aggregate initial endowments plus total profits; and (b) When there is no production, each consumer's wealth corresponds precisely to the value of her initial endowments. Particular cases of wealth functions satisfying (A.4) are those generated by a "fixed structure of profits" [given by: $r_i(\mathbf{p}, \bar{\mathbf{y}}) = \mathbf{p}\,\omega_i + a_i \sum_{j=1}^{n} \mathbf{p}\,\mathbf{y}_j$, where $a_i \geq 0$, and $\sum_{i=1}^{m} a_i = 1$], and by a "fixed structure of shares" (which can be defined as: $r_i(\mathbf{p}, \bar{\mathbf{y}}) = \mathbf{p}\,\omega_i + \sum_{j=1}^{n} \theta_{ij}\,\mathbf{p}\,\mathbf{y}_j$, with $\theta_{ij} \geq 0$, and $\sum_{i=1}^{m} \theta_{ij} = 1$, for all j).

Notice that assumptions (A.1), to (A.4) ensure that $\mathcal{A}(\omega)$ is nonempty (to see that simply let $\mathbf{y}_j = 0$, $\forall\ j$, and $\mathbf{x}_i = \omega_i$, $\forall\ i$), and compact.

Consider now the following definitions:

<u>Definition</u> <u>2.3.-</u> We shall say that $\phi_j : \mathbb{P} \times \mathfrak{F} \longrightarrow \mathbb{P}$ is a **Regular Pricing Rule**, if ϕ_j is an upper hemicontinuous correspondence, with nonempty, closed and convex values.

Definition 2.4.- We shall say that $\phi_j : \mathbb{P} \times \mathfrak{Y} \longrightarrow \mathbb{P}$ is a **Pricing Rule with Bounded Losses**, if a scalar $\alpha_j \leq 0$ exists such that, for each $(p, \bar{y})$ in $\mathbb{P} \times \mathfrak{Y}$,

$$q\, y_j \geq \alpha_j, \quad \forall \; q \in \phi_j(p, \bar{y})$$

Remark 2.2.- The combination of the notions of bounded-losses and regularity implies a non-trivial structure on the pricing rule. In particular, it prevents a firm from setting:

$$\phi_j(p, \bar{y}) \equiv \{ q^o \}$$

(constant) for all $(p, \bar{y})$ in $\mathbb{P} \times \mathfrak{Y}$ (which would easily destroy any possibility of equilibria). The reader is encouraged to think about the nature of this implication [Bonnisseau & Cornet (1988 a, Remark 2.6) will help].

For a given $p \in \mathbb{P}$, let $b_i(p)$ denote the minimum value of $p\, x_i$, with $x_i \in X_i$ (that is, b_i denotes the minimum worth at prices p of a feasible consumption bundle for the *ith* consumer). This is clearly a continuous function (by virtue of the maximum Theorem). The next definition incorporates a restriction on the distribution of wealth which provides us with a straightforward survival assumption.

Definition 2.5.- Let $\phi_1, \phi_2, \ldots, \phi_n$ stand for the *n* firms' pricing rules. We shall say that a wealth structure, $(r_1, \ldots, r_m)$ is **compatible with firms' behaviour**, if there exists an arbitrary small scalar $\delta > 0$, such that, for each consumer and for every production equilibrium $(\mathbf{p}, \bar{\mathbf{y}})$ in $\bigcap_{j=1}^{n} \phi_j(\mathbf{p}, \bar{\mathbf{y}}) \times \mathfrak{F}$,

$$r_i(\mathbf{p}, \bar{\mathbf{y}}) \geq b_i(\mathbf{p}) + \delta$$

Thus we say that the distribution of wealth is compatible with firms' behaviour if, in a production equilibrium, each consumer's budget set has a nonempty interior.

Definition 2.6.- We shall say that a price vector $\mathbf{p}^* \in \mathbb{P}$, and an allocation $[(\mathbf{x}_i^*), \bar{\mathbf{y}}^*]$, yield an **Equilibrium** if the following conditions are satisfied:

(α) For each $i = 1, 2, \ldots, m$, $\mathbf{x}_i^*$ maximizes u_i over the set of points $\mathbf{x}_i$ in X_i such that:

$$\mathbf{p}^* \mathbf{x}_i \leq r_i(\mathbf{p}^*, \bar{\mathbf{y}}^*)$$

(β) For every $j = 1, 2, \ldots, n$, the jth firm is in equilibrium, that is,

$$\mathbf{p}^* \in \bigcap_{j=1}^{n} \phi_j(\mathbf{p}^*, \bar{\mathbf{y}}^*)$$

(γ) $\sum_{i=1}^{m} \mathbf{x}_i^* - \sum_{j=1}^{n} \mathbf{y}_j^* \leq \omega$, and:

$$\sum_{i=1}^{m} \mathbf{x}_{it}^* - \sum_{j=1}^{n} \mathbf{y}_{jt}^* < \omega_t \implies p_t^* = 0$$

That is, an Equilibrium is a situation in which: (a) Consumers maximize their preferences subject to their budget constraints; (b) Every firm is in equilibrium; and (c) All markets clear.

Let $\mathbb{E}$ denote the class of economies just described, that is, market economies satisfying assumptions (A.1) to (A.4). Then we can show [see Villar (1993)]:

THEOREM 2.1.- Let E stand for an economy in $\mathbb{E}$. An Equilibrium exists when firms follow regular pricing rules with bounded losses, and the wealth structure is compatible with firms' behaviour.

The structure of the model (and the proof of the existence theorem) allows us to interpret the functioning of this economy as follows: (a) There is an auctioneer who calls out both a price vector (to be seen as proposed market prices), and a vector of efficient production plans. (b) Given these prices and production plans, the *ith* consumer chooses that consumption bundle which maximizes her utility subject to her wealth constraint. (c) Firms check out whether the proposed "prices-production" pair agrees with their objectives. When this is so, the price vector is a candidate for a market equilibrium. (d) When not all firms agree on the proposed prices-production combination, or markets do not clear, the auctioneer tries a new proposal. For that, she chooses those prices and production plans such that, they maximize the value of the "excess demand" and minimize the distance between each pricing rule and the proposed prices.

The next Sections are devoted to the analysis of particular pricing rules, which will convey substance to this abstract framework. We consider first the family of Loss-Free Pricing Rules, and then the Marginal Pricing Rule and other Regulation Policies. The efficiency problem will also be tackled.

References to the Literature.- There is a number of existence results which refer to abstract pricing rules, results that can then be particularized so as to encompass most of the pricing rules to be considered in next Sections. The papers by MacKinnon (1979) and Dierker, Guesnerie & Neuefeind (1985) are pioneering contributions to this area. Bonnisseau & Cornet (1988 a) provide an extremely general existence result, for the case in which firms' losses are bounded (this paper may be thought of as a benchmark in the literature on the existence of equilibria with non-convex technologies). Vohra (1988 a) presents an alternative existence result, using slightly different assumptions (and an easier proof). A degree theoretic existence result can be found in Kamiya (1988) (where the question of uniqueness is also analyzed). Simplified versions of Bonnisseau & Cornet's model appear in Villar (1991), (1994) where relatively easy existence proofs are provided. See also Bonnisseau (1988), for a discussion of some interconnections. The reader is encouraged to go through Brown's (1991) survey and Bonnisseau & Cornet (1988 a) paper in order to get a deeper review of the existence results.

3.- LOSS-FREE PRICING RULES: THE POSITIVE APPROACH.

Loss-Free Pricing is a family of pricing rules with bounded losses, where firms' equilibrium profits are nonnegative. This family covers most of the ways of modelling the behaviour of non-convex firms in a context of unregulated markets (what we referred to as Positive Models). Even though those regulation policies satisfying a break-even constraint formally also belong to this family, we shall discuss these models in connection with the Marginal Pricing Rule. In order to deal with Loss-Free pricing rules in a simpler way, we shall specialize our model, by focusing on the case of private ownership economies (those in which the income function corresponds to a "fixed structure of shares").

Let us formally introduce the family of loss-free pricing rules and the companion assumption:

Definition 3.1.- We shall say that $\phi_j : \mathbb{P} \times \mathfrak{F} \longrightarrow \mathbb{P}$ is a **Loss-Free Pricing Rule**, if for each $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathfrak{F}$, all $\mathbf{q}_j$ in $\phi_j(\mathbf{p}, \bar{\mathbf{y}})$, we have:

$$\mathbf{q}_j \mathbf{y}_j \geq 0$$

Thus, a firm is said to follow a Loss-Free pricing rule whenever it does not find "acceptable" any prices-production combination yielding negative profits.

We substitute now assumption (A.4) by the following:

A.4*.- $r_i(p, \bar{y}) = p\, \omega_i + \sum_{i=1}^{n} \theta_{ij}\, p\, y_j$ (with $\sum_{i=1}^{n} \omega_i = \omega$, $\sum_{j=1}^{m} \theta_{ij} = 1$).

Call now $\mathbb{E}^*$ to the set of private ownership market economies satisfying assumptions (A.1), (A.2), (A.3) and (A.4*). Observe that under assumption (A.4*), it follows from (A.3) that, for every $(p, \bar{y}) \in \mathbb{P} \times \mathfrak{F}$,

$$r_i(p, \bar{y}) \geq p\omega_i > b_i(p)$$

Therefore, since there is a finite number of consumers, one can take (see Definition 2.5):

$$\delta = \frac{1}{2} \operatorname{Min.}_{p} \left\{ \operatorname{Min.}_{i} [p\omega_i - b_i(p)] \right\}$$

and be sure that $r_i(p, \bar{y}) \geq b_i(p) + \delta$, $\forall\ (p, \bar{y}) \in \mathbb{P} \times \mathfrak{F}$. Then, the following result turns out to be an immediate consequence of Theorem 2.1:

Proposition 3.1.- Let E stand for an economy in $\mathbb{E}^*$. A Market Equilibrium exists when firms follow regular and loss-free pricing rules.

Thus, in the context of private ownership market economies which satisfy (A.1), (A.2), (A.3) and (A.4*), loss-free pricing rules constitute a special case of pricing rules for which the wealth structure is always compatible with firms' behaviour.

We shall now consider two prominent examples of regular and loss-free pricing rules: Profit Maximization (both constrained and unconstrained), and Average Cost Pricing. The reader may well consult Bonnisseau & Cornet (1988 a, Section 3), and Dehez & Drèze (1988 a, b) for details.

3.1.- Profit Maximization.

Let us now formalize different types of profit maximization in terms of pricing rules. We start by considering the convex case, which is the paradigm of profit maximization. When technologies are convex, (unconstrained) Profit Maximization can be defined in terms of the following pricing rule:

$$\phi_j^{PM}(p, \bar{y}) \equiv \{ q \in \mathbb{P} \ / \ q\, y_j \geq q\, y'_j \ , \ \forall\, y'_j \in Y_j \}$$

This pricing rule associates with every efficient production plan, the set of prices which support it as the most profitable one (that is, in this case ϕ_j coincides with the inverse supply mapping). Under our assumptions, this is obviously a *loss-free* pricing rule (since $0 \in Y_j$ for each j); it is also easy to deduce that ϕ_j^{PM} is *regular* (the maximum theorem implies the upper hemicontinuity, whilst the convexity of Y_j brings about the nonemptyness, and convexity follows trivially). Thus, in particular, the existence of a Competitive equilibrium is obtained as a Corollary.

Corollary 3.1.- Let $E \in \mathbb{E}^*$, and suppose that Y_j is convex, $\forall\, j$. Then a Competitive Equilibrium exists.

Remark 3.1.- Observe that if Y_j is a convex set, $j = 1, 2, \ldots, n$, by requiring that $Y_j \cap \mathbb{R}_+^{\ell} = \{ 0 \}$ in (A.1), assumption (A.2) can be replaced by the irreversibility hypothesis in Debreu (1959, p. 40).

We consider now a pricing rule which can be regarded as an extension of the profit maximization principle to nonconvex production sets. This pricing rule, termed Voluntary Trading, is a refinement of Output-Constrained Profit Maximization.

The notion of Voluntary Trading was introduced by Dehez & Drèze (1988 a) as a way of extending the notion of competitive equilibria to a context whereby firms behave as quantity takers, and there may be increasing returns to scale [see Dierker & Neuefeind (1988) for a different approach]. Let us first define the Output-Constrained Profit Maximization pricing rule as follows(3):

$$\phi_j^{OC}(\mathbf{p}, \bar{\mathbf{y}}) \equiv \{ \mathbf{q} \in \mathbb{R}_+^\ell \; / \; \mathbf{q}\,\mathbf{y}_j \geq \mathbf{q}\,\mathbf{y}, \;\; \forall\, \mathbf{y} \in Y_j \;\; \text{with } \mathbf{y} \leq \mathbf{y}_j^+ \;\}$$

(where $\mathbf{y}_j^+$ denotes a vector in $\mathbb{R}_+^\ell$ with coordinates max. $\{ 0, y_{jh} \}$, for $h = 1, 2, \ldots, \ell$). The main feature of this pricing rule is that at those prices "it is not more profitable for the producers to produce less. Thus at an equilibrium, producers maximize profits subject to a sales constraint" [Cf. Dehez & Drèze (1988 a, p. 210)].

Dehez & Drèze suggest the following refinement of Output-Constrained Profit Maximization, which describes the minimality of the output prices:

$$\psi^*(\mathbf{p}, \bar{\mathbf{y}}) \equiv \{ \mathbf{q} \in \phi_j^{OC}(\mathbf{p}, \bar{\mathbf{y}}) \; / \; \text{there is no } \mathbf{q}' \in \phi_j^{OC}(\mathbf{p}, \bar{\mathbf{y}}),\; \mathbf{q}' < \mathbf{q}, \;\; \text{and } q'_j = q_j\,, \;\; \text{for } h \in I_j(y_j) \}$$

where $I_j(y_j)$ denotes the commodities which are inputs for the *jth* firm.

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(3) Note that we define this pricing rule as a mapping from $\mathbb{P} \times \mathfrak{Y}$ into $\mathbb{R}^\ell$, rather than into $\mathbb{P}$, for reasons which will be apparent below.

This condition of minimal output prices says that lower output prices cannot sustain the same output quantities.

For each $(\mathbf{p}, \bar{\mathbf{y}}) \in \mathbb{P} \times \mathfrak{F}$, Voluntary Trading, denoted by $\phi_j^{VT}(\mathbf{p}, \bar{\mathbf{y}})$, can be defined as the smallest closed[4] and convex-valued correspondence containing $\psi^*(\mathbf{p}, \bar{\mathbf{y}}) \cap \mathbb{P}$.

Observe that since $0 \in Y_j$ [assumption (A.1)], ϕ_j^{VT} is a loss-free pricing rule. Furthermore, it is convex and compact valued by definition (and hence upper hemicontinuous); it is also easy to see that it is nonempty valued. Thus, under assumptions (A.1), (A.2), (A.3) and (A.4*), ϕ_j^{VT} is a loss-free and regular pricing rule. Hence:

Corollary 3.2.- Let $E \in \mathbb{E}^*$. Then an equilibrium exists when firms follow Voluntary Trading.

Dehez & Drèze (1988 a, Th. 1) show that Voluntary Trading coincides with (unconstrained) Profit Maximization when production sets are convex.

Remark 3.2.- Dehez & Drèze (1988 a) call Voluntary Trading to ϕ_j^{OC}, while ϕ_j^{VT} is left unchristened. We have preferred to apply the most characteristic name to their refined pricing rule. Let us also point out that they define this pricing rule in a more basic (although actually equivalent) way.

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(4) A correspondence $\Gamma: D \longrightarrow \mathbb{R}^k$ is said to be closed-valued if $\Gamma(\mathbf{z})$ is a closed set, for each $\mathbf{z} \in D$. Γ is said to be closed if it has a closed graph. Let us recall here that if Γ is closed and has compact values, then it is upper hemicontinuous.

Remark 3.3.- Scarf (1986) develops a model of Input-Constrained Profit Maximization in his analysis of economies with increasing returns and nonempty cores. This model is presented in Section 5.3.

3.2.- Average-Cost Pricing.

Average cost-pricing is a pricing rule with a long tradition in economics, both in positive and normative analysis. We shall concentrate here on the positive approach, and leave the normative one for the next Section.

The Average Cost Pricing Rule can be formulated as follows[5]:

$$\phi_j^{AC}(p, \bar{y}) \equiv \{ q \in \mathbb{P} \;/\; q\, y_j = 0 \}$$

that is, this rule associates with every efficient production plan for the *jth* firm, those prices yielding null profits.

Under assumptions (A.1), (A.2), (A.3) and (A.4*), ϕ_j^{AC} is obviously a loss-free and regular pricing rule. Hence, Proposition 3.1 provides an implicit existence result for those economies where firms are instructed to obtain zero profits. Formally:

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(5) This way of defining the average cost pricing rule places no restriction at the origin (i.e., for $\bar{y}$ such that $y_j = 0$, $\phi_j^{AC}(p, \bar{y}) \equiv \mathbb{P}$). It is then customary to define ϕ_j^{AC} as the closed convex hull of the following set:

$$\{ q \in \mathbb{R}_+^{\ell} \;/\; \exists \{ q^{\nu}, y_j^{\nu} \} \subset \mathbb{P} \times [\mathfrak{F}_j \setminus 0] \text{, such that, } \{ q^{\nu}, y_j^{\nu}\} \longrightarrow (q, 0), \text{ with } q^{\nu} y_j^{\nu} = 0 \}$$

Corollary 3.4.- Let $E \in \mathbb{E}^*$. Then an Average Cost Pricing Equilibrium does exist.

When production sets are convex cones (constant returns to scale), Average Cost Pricing coincides with Profit Maximization (and hence with Voluntary Trading). Yet in general Average Cost Pricing may well be inconsistent with profit maximization (either constrained or unconstrained). This implies that this pricing rule belongs to a family of loss-free and regular pricing rules whose associated equilibria may be difficult to sustain, since some firms may find it profitable to deviate from the equilibrium production plans. This may happen both for the case of decreasing returns and for the case of increasing returns to scale. One may thus consider whether there exists some restriction on production sets that makes average cost pricing compatible with constrained profit maximization (this would extend the properties of convex cones to a more general setting).

Following an idea in Scarf (1986), Dehez & Drèze (1988 b) define the notion of Output Distributive Sets, as a way of characterizing those production sets for which Average Cost Pricing is compatible with Voluntary Trading. In order to define the notion of distributive sets, (part of) the input-output configuration must be fixed a priori. Thus let us start by considering the following assumption:

A.0.- For every j, the set $\mathcal{L} \equiv \{ 1, 2, \ldots, \ell \}$ can be partitioned into two disjoint subsets, O_j and its complement, so that if $y_j \in Y_j$ and $t \in O_j$, then $y_{jt} \geq 0$. We shall refer to goods in O_j as outputs, and write production plans as $y_j = (a_j, b_j)$, with $b_j \geq 0$, in the understanding that b_j is a point in the subspace of outputs.

Assumption (A.0) simply says that we can distinguish a priori two groups of commodities: outputs (which are positive), and other commodities. The way of presenting this idea tries to emphasize that there is flexibility in the way of taking the partition.

We shall give now the definition of output distributive sets, assuming that (A.0) holds. Before that, let us introduce a piece of notation. For any finite collection $(y_j^1, y_j^2, \ldots, y_j^k)$ of production plans, $y_j^t \in Y_j$ for all t, we shall denote by $K(y_j^1, y_j^2, \ldots, y_j^k)$ the convex cone with vertex zero generated by points $(y_j^1, y_j^2, \ldots, y_j^k)$, that is,

$$K(y_j^1, y_j^2, \ldots, y_j^k) \equiv \{ y \in \mathbb{R}^\ell \ / \ y = \sum_{t=1}^{k} \alpha^t y_j^t, \ \alpha^t \geq 0 \}$$

<u>Definition</u> <u>3.2.-</u> We shall say that Y_j is an **Output-Distributive Set** whenever, for any finite collection $(y_j^1, y_j^2, \ldots, y_j^k)$ of production plans, with $y_j^t = (a_j^t, b_j^t)$ and $y_j^t \in Y_j$ for all t, the following inclusion is satisfied:

$$K(y_j^1, \ldots, y_j^k) \cap \{ y \in \mathbb{R}^\ell \ / \ y_j = (a_j, b_j) \ \& \ b_j \geq b_j^t \ \ \forall t \} \subset Y_j$$

Thus a production set is said to be Output-Distributive if any (nonnegative) weighted sum of feasible production plans is feasible if it involves more outputs than any of the original plans.

Under assumption (A.0), the following properties are easily deduced from the definition, and qualify the notion of distributivity:

- The definition implies nondecreasing returns to scale (i.e., if y_j is in Y_j, λy_j will also be in Y_j, for any scalar $\lambda \geq 1$). In particular they cover the case of constant returns to scale.

- If Y is distributive then Y is additive (that is, $y_j + y'_j \in Y_j$ whenever $y_j, y'_j \in Y_j$).

- If Y_j is an output-distributive set, the "isoquants"

$$A_j(b_j) = \{ a_j \ / \ (a_j, b_j) \in Y_j \}$$

are non-empty, closed and convex sets.

The next Proposition gives us the essential property which allows for the compatibility of constrained profit maximization and average cost pricing [see Scarf (1986, Th. 1), Dehez & Drèze (1988 b, Prop. 1)]:

Proposition 3.2.- Let Y_j be an Output-Distributive Set satisfying assumptions (A.0) and (A.1). Then for every $y'_j \in \mathfrak{F}_j$ there exists $p' \in \mathbb{R}^{\ell}$, $p' \neq 0$, such that:

$$0 = p'y' \geq p'y$$

for all $y_j = (a_j, b_j)$ such that $b_j \leq b'_j$.

Let us define now the constrained profit maximization pricing rule compatible with average cost pricing:

$$\psi^{DD} \equiv \phi^{VT} \cap \phi^{AC}$$

This pricing rule exhibits some competitive features: It implies that, in equilibrium, firms behave as profit maximizers with quantity constraints, and they just do break even.

Observe that ψ^{DD} is defined as the intersection of two regular and loss-free pricing rules; therefore it is also an upper hemicontinuous correspondence, with convex and closed values, satisfying the loss-free property. Proposition 3.2 ensures that such intersection is actually nonempty. Thus, ψ^{DD} is a regular and loss-free pricing rule, so that the next Corollary is an immediate consequence of Propositions 3.1 and 3.2 (and provides us with an answer to the question about the compatibility between constrained profit maximization and average cost pricing, in nonconvex environments):

Corollary 3.5.- Let E be an economy in $\mathbb{E}^*$ satisfying assumption (A.0), and let Y_j be an Output-Distributive set. Then, ψ^{DD} is a regular pricing rule, so that an equilibrium where firms follow the ψ^{DD} pricing rule, does exist.

Remark 3.4.- It is worth stressing that distributivity essentially characterizes the compatibility between constrained profit maximization and average cost pricing (that is, the converse of Proposition 3.2 is also true, provided some technicalities are taken into account). See Scarf (1986) and Dehez & Drèze (1988 b).

References to the Literature.- Note that there are few cases in which nonconvexities are compatible, at least partially, with the classical competitive model. This happens when nonconvexities are small in relation to the size of the economy (in this case one can still get an upper-hemicontinuous aggregate supply mapping), and when they correspond to increasing returns due to external economies [see Arrow & Hahn (1971, Ch.6), Mas-Colell (1987, Ch. VI)].

When we abandon the specific context described above, an alternative definition of firms' behaviour is required. Monopolistic (or oligopolistic) competition arises as a natural framework to deal with non-convex firms: if there are increasing returns to scale firms will not be negligible and thus will not behave as price-takers. Following Negishi's (1961) model, Arrow & Hahn (1971, 6.4) present a model of monopolistic competition in which no assumption is made about the convexity of production sets. Silvestre (1977), (1978) criticizes this model and offers alternatives with better foundations. Nevertheless, these models are still restrictive. A related line of research (using "objective" rather than "subjective" demand curves) was developed by Gabsewicz & Vial (1972) and Fitzroy (1974); these models turn out to be even more restrictive (since objective demands impose much more structure than subjective ones). Indeed, the possibility of extending partial equilibrium results to a general equilibrium framework, in the realm of imperfect competition, faces enormous difficulties even with convex production sets [see the analysis in Roberts & Sonnenschein (1977), and the recent survey in Benassy (1991)].

An alternative approach to modelling the behaviour of non-convex firms, compatible with both positive and normative viewpoints, consists of allowing for the presence of quantity constraints (due to input or demand restrictions, or the existence of quantitative targets). In this context (constrained) profit maximization may be well defined. The existence of general equilibrium with quantity constraints was first dealt with in the classic paper by Scarf (1986) (where he analyzed the non-emptyness of the core in an economy with increasing returns). Besides the model by Dehez & Drèze (1988 a), already discussed, let us also refer here to that one by Dierker & Neuefeind (1988) which extends the results in Dierker, Guesnerie & Neuefeind (1985), allowing for the presence of quantity targets.

Existence results for Average Cost Pricing (or, more generally, Mark-up Pricing) also abound. Apart from those presented above, let us recall here that Dierker, Guesnerie & Neuefeind (1985) prove the existence of equilibrium when firms follow several forms of Average Cost Pricing. Böhm (1986) and Corchón (1988) develop models where firms set prices by adding a mark-up over average costs. Herrero & Villar (1988), and Villar (1991) provide average cost pricing models where the production side is formulated as a nonliear Leontief (resp. a nonlinear von Neumann) model.

4. THE NORMATIVE APPROACH: MARGINAL PRICING AND OTHER REGULATION POLICIES.

Let us focus now on Normative Models (that is, pricing rules which may be interpreted as regulation policies for public utilities under non-convex production sets). We shall consider first the pricing rule from which most of the existence results on general equilibrium in nonconvex environments originated: the Marginal Pricing Rule[6]. This pricing rule shares with Voluntary Trading the feature that it coincides with profit maximization when production sets are convex. We shall move then towards other regulation policies which satisfy a break-even constraint (and hence may be regarded as refinements of Average Cost Pricing). Three of these pricing rules will be considered: Two-Part Marginal Pricing, Boiteaux-Ramsey prices and Aumann-Shapley values.

4.1.- The Notion of Marginal pricing.

Consider the case in which resources are to be allocated through a market mechanism, and suppose that production sets are assumed to be closed and satisfy free-disposal (that is, $Y_j - \mathbb{R}^{\ell}_{+} \subset Y_j$). Then, irrespective of the convexity assumption, a general principle for

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[6] The expression "Marginal Pricing", instead of the usual "Marginal Cost Pricing" is used in order to remind that in the absence of convexity (more precisely, in the absence of convexity of the iso-outputs sets), this pricing rule may not imply cost minimization. See the discussion in Guesnerie (1990, Section 5.2).

achieving Pareto optimality is that prices must equal the marginal rates of transformation (both for consumers and firms). If this were not so, it would be possible to reallocate commodities so that someone would be better-off.

When production sets have a smooth (i.e. differentiable) boundary, marginal rates of transformation are well defined, and marginal prices coincide with the vector of partial derivatives at every efficient production plan. When production sets are convex (but may not have a smooth boundary), one has to take a generalized view of what marginal rates of transformation are. In particular, marginal prices can be associated with the cone of normals which is defined as follows: Let A be a convex subset of $\mathbb{R}^{\ell}$; the <u>Normal</u> <u>Cone</u> of A at s, $\mathbb{N}(A, s)$, is given by:

$$\mathbb{N}(A, s) \equiv \{ p \in \mathbb{R}^{\ell} \;/\; p(y - s) \leq 0 , \;\; \forall\, y \in A \}$$

Thus when production sets are convex, marginal pricing implies profit maximization at given prices.

When production sets are neither convex nor smooth, we need a way of extending still further the notion of "marginal rates of transformation". There are several alternatives for that [see Kahn & Vohra (1987, Section 2), Cornet (1990, Appendix) for a discussion], but the nowadays standard definition is based on Clarke's normal cones. In order to define Clarke's normal cone, let us start with the following definition:

Definition 4.1.- Let C be a closed subset of $\mathbb{R}^{\ell}$ and $\mathbf{x} \in C$. A vector $\mathbf{v}$ is **orthogonal** to C at **x**, denoted $\mathbf{v} \perp C(\mathbf{x})$, if[7]

$$dist[(\mathbf{v}+\mathbf{x}),\ C] = ||\ \mathbf{v}\ ||$$

If **x** is a point in the interior of C, then $\mathbf{v} = \mathbf{0}$ is the only vector orthogonal to C at **x** (indeed **0** is orthogonal to any point in C). Thus the points of interest are the points in the boundary.

Clarke's normal cone $\mathbb{N}(Y, \mathbf{y})$ is then defined as follows:

Definition 4.2.- Let C be a closed subset of $\mathbb{R}^{\ell}$ and $\mathbf{x} \in C$. Then, the **Normal Cone** $\mathbb{N}(C, \mathbf{x})$ [in the sense of Clarke (1975)] to C at **x** is the closed convex hull of the set:

$$\{\ \mathbf{v} \in \mathbb{R}^{\ell}\ /\ \ \mathbf{v} = \lambda \lim . \frac{\mathbf{v}_i}{||\mathbf{v}_i||}\ ,\quad \lambda \geq 0,\ \ \mathbf{v}_i \perp C(\mathbf{x}_i),\ \mathbf{x}_i \longrightarrow \mathbf{x},\ \mathbf{v}_i \longrightarrow 0\ \}$$

By this definition the Clarke Normal Cone at a point **x** is the convex cone generated by the vectors orthogonal to C at **x**, and the limits of vectors which are orthogonal to C in a neighbourhood of **x** [Cf. Quinzii (1992, p. 19)].

Let Y_j be a production set, and $\mathbf{y}_j$ a boundary point. We can define now Marginal Pricing as follows:

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(7) *dist[.]* denotes the euclidean distance, while $||.||$ stands for norm.

$$\phi_j^{MP}(p, \bar{y}) \equiv \mathbb{N}(Y_j, y_j) \cap \mathbb{P}$$

where $\mathbb{N}(Y_j, y_j)$ denotes Clarke's normal cone to Y_j at y_j.

The following properties of normal cones are most useful [see Clarke (1983, Ch. 2), Cornet (1990, Lemma 4)]:

Proposition 4.1.- Let Y be a closed subset of $\mathbb{R}^{\ell}$, and let $y \in Y$. Then:

(i) If $py \geq py'$ for all $y' \in Y$, then $p \in \mathbb{N}(Y, y)$. In case Y is convex, the converse is also true, i.e., if $p \in \mathbb{N}(Y, y)$ then $py \geq py'$, for all $y' \in Y$.

(ii) If $Y - \mathbb{R}^{\ell}_{+} \subset Y$, then the following conditions hold:

(ii, a) $\mathbb{N}(Y, y) \subset \mathbb{R}^{\ell}_{+}$ for every $y \in Y$.

(ii, b) The correspondence $\mathbb{N}(Y, .)$ from Y to $\mathbb{R}^{\ell}$ is closed.

4.2.- The Existence of Marginal Pricing Equilibrium.

We shall present here an existence result for Marginal Pricing which derives from Theorem 2.1. Other existence results, dispensing with the bounded-losses assumption, are available in the literature [see for instance Bonnisseau & Cornet (1990 a, b), Vohra (1992), and the references below].

Let $\mathbf{e}$ stand for the unit vector, $\mathbf{e} \equiv (1, 1, \ldots, 1)$. The next Proposition [corresponding to Bonnisseau & Cornet (1988, Lemma 4.2)], provides us with sufficient conditions for marginal pricing to be a pricing rule with bounded losses.

Proposition 4.2.- Let $Y_j \subset \mathbb{R}^\ell$ be nonempty and closed, with $Y_j - \mathbb{R}^\ell_+ \subset Y_j$.

(i) Let α_j be a real number. The two following conditions are equivalent:

(i,a) $\forall\ (y_j, p) \in \mathfrak{F}_j \times \{\ \mathbb{N}(Y_j, y_j) \cap \mathbb{P}\ \}$, one has $py_j \geq \alpha_j$.

(i,b) $\forall\ y_j \in Y_j$ one has $[\alpha_j e, y_j] \subset Y_j$ (i.e., Y_j is star shaped).

(ii) Conditions (i,a), (i,b) are satisfied if there exists a non-empty, compact subset K_j of $\mathbb{R}^\ell$, if

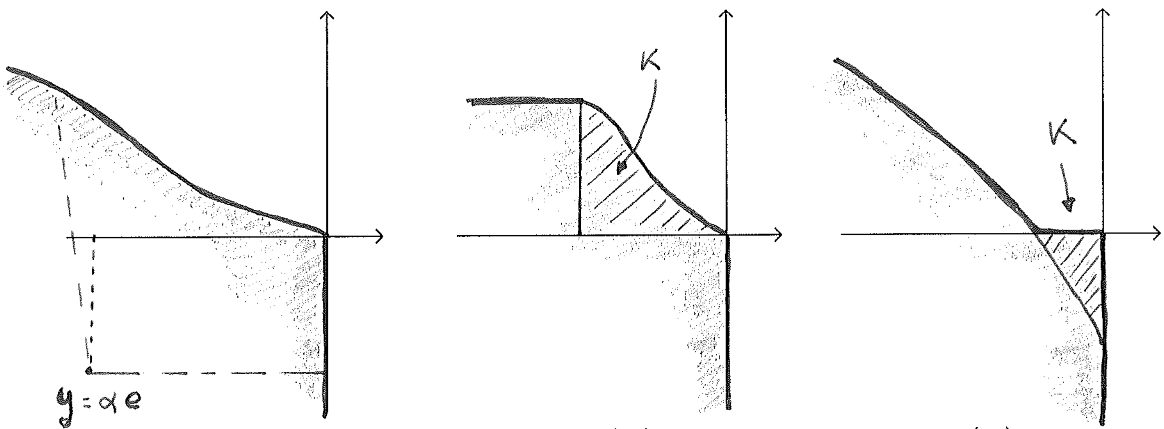
$$\alpha_j = \inf_h \{\ y_{jh}\ /\ y_j = (y_{jh}) \in K_j\ \}$$

and if one of the following conditions holds:

(C.1) $Y_j = K_j - \mathbb{R}^\ell_+$, or

(C.2) $Y_j \backslash K_j$ is convex.

Figure 4.5 illustrates the case of production sets satisfying the properties in Proposition 4.2.



(a) Y_j star-shaped

(b) $Y_j = K_j - \mathbb{R}^2_+$

(c) $Y_j \setminus K_j = C_j$ convex

- Figure 4.1 -

The following result is an immediate consequence of Propositions 4.1, 4.2 and Theorem 2.1:

Corollary 4.1.- **Under assumptions (A.1) to (A.4), suppose that Y_j is star shaped for every *j*. Then, a Marginal Pricing Equilibrium exists when the income distribution satisfies the compatibility requirement.**

There are two properties of Marginal Pricing worth considering:

(i) This pricing rule satisfies the necessary conditions for optimality, and it coincides with profit maximization when production sets are convex. Yet, when production sets are not convex, these necessary conditions may well not be sufficient (see the next Section).

(ii) When there are increasing returns to scale, Marginal Pricing implies losses ("marginal costs" are smaller than "average costs"). This entails that this pricing rule requires the design of a system of transfers (embodied in consumers' wealth functions, say), so that these firms can cover their losses. Letting aside the informational problem [see Calsamiglia (1977)], this can be seen as an additional complication of the regulation policy which requires taking decisions about its distributional impact. In particular, a Marginal Pricing Equilibrium may not be Individually Rational; even if it is Individually Rational, some consumers may feel that they are paying "too much", so that there is little hope for social stability (these equilibria will typically fail to be in the core).

4.3.- Two-Part Marginal Pricing and other Regulation Policies.

The distributional problems associated with the use of marginal pricing, induced the consideration of regulation policies which satisfy a break-even constraint. Observe that in the case of multiproduct non-convex firms, the break-even constraint is compatible with a number of ways of pricing the regulated firms' commodities. We shall comment here on three regulation policies which satisfy such a constraint: Two-Part Marginal Pricing, Boiteaux-Ramsey Prices and Aumann-Shapley Values.

4.3.1.- Two-Part Marginal Pricing.

A closer look at the necessary conditions for optimality shows that what is essentially required is that consumers equate marginal prices to the marginal quantities they demand. This suggests that using a (personalized) system of non-linear prices one can meet both the necessary conditions for optimality and the break-even constraint. This is the essence of two-part tariffs, where consumers who buy positive amounts of the goods produced by non-convex firms are charged an entrance fee plus a proportional one.

Let us briefly describe now the Two-Part Marginal pricing model developed by Brown, Heller & Starr (1992). They consider a general equilibrium model of a private ownership economy with a single non-convex firm, indexed as firm 0 (a regulated monopoly), and n competitive convex firms. The non-convex firm produces a single output, good 0 (the monopoly good), which is not produced by any other firm, and the initial endowment of this good is taken to be zero. Regulation takes the form of marginal pricing with personalized "hook-up" fees charged for the right to consume the monopoly good. The hook-up fees are intended to recover the losses incurred by the monopoly when using marginal pricing. Hence in equilibrium the monopoly makes zero profits.

This pricing policy implies that the *ith* consumer's budget constraint will exhibit the following structure:

$$px_i \leq \begin{cases} p\omega_i + \sum_{j=0}^{n} \theta_{ij}\, py_j & \text{if } x_{i0} = 0 \\ p\omega_i + \sum_{j=0}^{n} \theta_{ij}\, py_j - q_i & \text{otherwise} \end{cases}$$

where q_i represents the *ith* consumer's hook-up fee (that she only pays when consuming positive amounts of the monopoly good). The restriction on these q_i is that $\sum q_i = \min.(0, -py_0)$.

Brown, Heller & Starr (1992) show that there exists an equilibrium for this economy, where the monopoly is regulated according to the Two-Part Marginal pricing rule. The basic idea underlying their existence proof is the notion of willingness to pay and the assumption that, in equilibrium, the aggregate willingness to pay exceeds the losses of the regulated monopoly. They also show that one can choose the hook-up fee in a way such that the equilibria are individually rational.

Assuming that the set of attainable consumptions is compact for each i, let X_i^* denote a convex and compact subset of $\mathbb{R}^\ell$ containing in its interior the *ith* consumer's set of attainable consumptions. Besides the assumptions in Section 2, suppose that u_i is strictly quasi-concave, and let $r_i(\mathbf{p}, \bar{\mathbf{y}}) = \mathbf{p}\omega_i + \sum_{j=0}^{n} \theta_{ij} \mathbf{p}\mathbf{y}_j$. We can then calculate each household's "reservation level of utility", i.e., the maximum utility level she could obtain if the monopoly good were not available, $V_i(\mathbf{p}, \bar{\mathbf{y}})$ as the solution to the following program:

$$\begin{aligned} & \text{Max. } u_i(\mathbf{x}_i) \\ \text{s.t.} & \\ & \mathbf{p}\ \mathbf{x}_i \leq r_i(\mathbf{p}, \bar{\mathbf{y}}) \\ & x_{i0} = 0 \\ & \mathbf{x}_i \in X_i^* \end{aligned}$$

We can use now the expenditure function to calculate the income which is necessary to reach the reservation utility level, when the monopoly good is available. This income, $E_i[\mathbf{p}, V_i(\mathbf{p}, \bar{\mathbf{y}})]$, is given by the solution to:

$$\begin{aligned} & \text{Min. } \mathbf{p}\ \mathbf{x}_i \\ \text{s.t.} & \\ & u_i(\mathbf{x}_i) \geq V_i(\mathbf{p}, \bar{\mathbf{y}}) \\ & \mathbf{x}_i \in X_i^* \end{aligned}$$

Then, each consumer's willingness to pay for the monopolist's output, at $(\mathbf{p}, \bar{\mathbf{y}})$ is given by:

$$s_i(\mathbf{p}, \bar{\mathbf{y}}) = r_i(\mathbf{p}, \bar{\mathbf{y}}) - E_i[\mathbf{p}, V_i(\mathbf{p}, \bar{\mathbf{y}})]$$

that is, "it is the amount at current prices that must be subtracted from current income to reduce utility to what it was when the monopoly good was unavailable... Of course, s_i is an ordinal concept, i.e., it is independent of the utility representation" [Cf. Brown, Heller & Starr (1992, p. 62)].

The key assumption in Brown, Heller & Starr's model is that, at every production equilibrium $(\mathbf{p}, \bar{y})$, the aggregate willingness to pay exceeds the monopoly losses: $\sum_{i=1}^{m} s_i(\mathbf{p}, \bar{y}) > -\min(\mathbf{p}y_0, 0)$. They define then a <u>Proportional</u> <u>hook-up</u> <u>rule</u> as follows: Let $s(\mathbf{p}, \bar{y}) = \sum_{i=1}^{m} s_i(\mathbf{p}, \bar{y})$, and

$$\tau(\mathbf{p}, \bar{y}) = \frac{-\min(\mathbf{p}y_0, 0)}{s(\mathbf{p}, \bar{y})}$$

Observe that τ is well defined over production equilibria, since we are assuming that $s(\mathbf{p}, \bar{y}) > 0$ in that case. Hence the Proportional hook-up rule is given by:

$$q_i(\mathbf{p}, \bar{y}) \equiv \tau(\mathbf{p}, \bar{y})\, s_i(\mathbf{p}, \bar{y})$$

that is, the proportional hook-up charge for the *ith* consumer is a fraction of her willingness to pay.

The assumptions of the model imply that q_i is a continuous function of $(\mathbf{p}, \bar{y})$ over the set of production equilibria, such that: a) It is always non-negative and smaller than $s_i(\mathbf{p}, \bar{y})$ when this is a positive number; and b) It is equal to zero if $s_i = 0$. This implies that demands are continuous over production equilibria. Then, using Tietze extension theorem, and applying a modification of the Beato & Mas-Colell (1985) existence argument, Brown, Heller & Starr prove the existence of a Two-Part Marginal Pricing equilibrium with proportional hook-up fees.

4.3.2.- Other Regulation Policies.

Ramsey (1927) (for a single agent economy) and Boiteaux (1956) analyzed the necessary conditions for optimality subject to a break-even constraint. The prices which satisfy these conditions are called Boiteaux-Ramsey prices. In the simplest version, where there is a single non-convex firm producing two outputs b_1, b_2 (whose cross elasticities of demand can be neglected), the firm is required to balance its budget and price the outputs at q_1, q_2 according to the "inverse of elasticity" formula:

$$q_1 - c_1 = K \frac{b_1}{\partial x_1} \qquad q_2 - c_2 = K \frac{b_2}{\partial x_2}$$

where c_1, c_2 represent the marginal cost of producing goods 1 and 2, respectively, and ∂x_1, ∂x_2 are the partial derivatives of compensated demand of the two goods taken with respect to their corresponding own prices. The number K is determined by the budget equation:

$$C(\mathbf{b}) = q_1 b_1 + q_2 b_2 = c_1 b_1 + c_2 b_2 + K\, b_1^2 (\partial x_1)^{-1} + K\, b_2^2 (\partial x_2)^{-1}$$

where $C(\mathbf{b})$ is the given total cost of producing output **b** [Cf. Dierker, Guesnerie & Neuefeind (1985, pp. 1381-1382)].

The intuition behind this rule is that one has to charge relatively higher prices over those products whose demand is relatively more inelastic. It is worth noticing that these prices are obtained from conditions over the maximization of the aggregate surplus, and that their distributive effects may well run in any direction [Cf. Mas-Colell (1987, p. 55)].

A different pricing principle emerges from an axiomatization of cost allocation schemes inspired in the Shapley Value for non-atomic games [first analyzed in Aumann & Shapley (1974), and used in Billera, Heath & Raanan (1978) for telephone billing rates which share the cost of a telephone system]. As in the case of Marginal Pricing (and unlike Boiteaux-Ramsey pricing), these prices only depend on the cost of production, and allow to take advantage of the fact that the Shapley Value can be defined by an explicit formula. Interestingly enough, they can be characterized by a set of axioms on the cost functions and the quantities produced [see for instance Billera & Heath (1982), Mirman & Tauman (1982), Samet & Tauman (1982)].

In order to present these ideas, we shall follow closely the work in Mirman & Tauman (1982). Think of a firm producing r outputs, and let $\mathbb{F}$ be a family of functions ℓ defined on a full dimensional comprehensive subset $C^{\ell} \subset \mathbb{R}^r_+$, and such that $\ell(0) = 0$ (no fixed cost), and ℓ is continuously differentiable on C^{ℓ}. We define a price mechanism as a function $P:\mathbb{F} \times C^{\ell} \longrightarrow \mathbb{R}^r$ that, for each $\ell \in \mathbb{F}$, and for every $\mathbf{b} \in C^{\ell}$, assigns a vector of prices:

$$P(\ell, \mathbf{b}) = [P_1(\ell, \mathbf{b}), P_2(\ell, \mathbf{b}), \ldots, P_r(\ell, \mathbf{b})]$$

Here ℓ is to be interpreted as the cost function, and $\mathbf{b}$ as the output vector. A price mechanism is then a way of pricing the outputs as a function of quantities and costs.

Consider now the following axioms:

(CS) (COST-SHARING)

For every $f \in \mathbb{F}$ and every $\mathbf{b} \in C^{f}$: $\mathbf{b}\, P(f, \mathbf{b}) = f(\mathbf{b})$ (that is, total cost equals total revenue).

(A) (ADDITIVITY)

Let f, g, $h \in \mathbb{F}$ defined over the same domain, and such that $f = g + h$. Then: $P(f, \mathbf{b}) = P(g, \mathbf{b}) + P(h, \mathbf{b})$ (i.e., if the cost f can be broken into two components, g and h, then calculating the price determined by the cost function f can be accomplished by adding the prices determined by g and h separately).

(P) (POSITIVITY)

If f is non-decreasing on C^{f}, then $P(f, \mathbf{b}) \geq 0$

(C) (CONSISTENCY)

For $f \in \mathbb{F}$, Let $C = \{ z \in \mathbb{R}_+ \;/\; z = \sum_{t=1}^{r} b_t,$ for $\mathbf{b} \in C^{f} \}$. If there is a function G defined on C such that $f(\mathbf{b}) = G(\sum_{t=1}^{r} b_t)$, then:

$$P_i(f, \mathbf{b}) = P(G, \sum_{t=1}^{r} b_t)$$

(i.e., splitting commodities in irrelevant classifications -that is, in a way that does not affect costs-, has no effect on prices).

(R) (RESCALING)

Let $\ell \in \mathbb{F}$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a vector of r positive real numbers. Define $C^{\ell}(\lambda) = \{ \mathbf{z} \in \mathbb{R}^r_+ \ / \ z_t = b_t/\lambda_t, \ \text{for } \mathbf{b} \in C^{\ell} \}$, and let $g \in \mathbb{F}$ be a function on $C^{\ell}(\lambda)$ defined by $g(\mathbf{b}) = \ell(\lambda_1 b_1, \ldots, \lambda_r b_r)$. Then: $P_i(g, \mathbf{b}) = \lambda_i \ P[\ell, (\lambda_1 b_1, \ldots, \lambda_r b_r)]$ (i.e., changing the scale of a commodity yields an equivalent change in prices).

Mirman & Tauman show that there exists one and only one price mechanism P(.,.) satisfying the above five axioms, and that this mechanism is the Aumann-Shapley price mechanism, that is, that one defined through the formula:

$$P_i(\ell, \mathbf{b}) = \int_0^1 \frac{\partial \ \ell}{\partial \ b_i} (t\mathbf{b}) \ dt$$

The apparent connection between Aumann-Shapley values and Marginal prices is analyzed in Samet & Tauman (1982). They show that dropping the Cost-Sharing assumption (CS), and substituting axiom (P) by the following:

(P*) If ℓ is non-decreasing in a neighbourghood of $\mathbf{b}$, then $P(\ell, \mathbf{b}) \geq 0$

then, axioms (A), (P*), (C) and (R) actually characterize the Marginal Pricing rule.

Dierker, Guesnerie & Neuefeind (1985) provide an existence result for a family of Average Cost Pricing rules which includes Boiteaux-Ramsey and Aumann-Shapley pricing. Indeed, it can be shown that, under reasonable conditions, these pricing rules are regular, so that Theorem 2.1 applies.

References to the Literature.- The first results on the existence of equilibria with nonconvex firms refer to Marginal Pricing (with the exception of Scarf's (1986) paper, which was written in 1963, and some of those referred to in Section 3). The idea of regulating nonconvex firms by setting prices equal to marginal costs is an old wisdom (which can be associated to the names of Dupuit, Marshall, Pigou, Lerner, Allais and Hotelling among others). It derives from the observation that a necessary condition for Pareto optimality is that all agents equate prices to their marginal rates of transformation.

Mantel (1979) and Beato (1982) independently showed the existence of equilibrium in an economy with a single firm whose production set has a smooth boundary, but need not be convex. They realize that under the free disposal assumption, the set of efficient and attainable productions can be made homeomorphic to a simplex, and hence the nonconvexity can be handled in the convex "mirror's image".

Cornet (1990) (a paper written in 1982) provides a first existence theorem for marginal pricing in an economy with a single firm but dispensing with the smoothness assumption. For that he introduces Clarke's normal cones as the proper way of defining marginal pricing in the general case.

Brown & Heal (1982) gave an index-theoretic proof of existence for Mantel's model. Beato & Mas-Colell (1985) extend the existence result for the case of several non-convex firms, and Brown, Heal, Khan & Vohra (1986) analyze the case of a private ownership economy with a single non-convex firm and several convex firms. More general results on this specific pricing rule appear in Bonnisseau & Cornet (1990 a, b), and Vohra (1992). See also the problem raised in Jouini's (1988) paper.

The existence of two-part marginal cost pricing equilibrium is established in Brown, Heller & Starr (1992), in a model described above.

There is a number of contributions which analyze different pricing policies in terms of the properties of the associated cost-functions. Besides those already referred to, let us mention the works of Mirman, Samet & Tauman (1983), Greenberg & Shitovitz (1984), Mirman, Tauman & Zang (1985), (1986), Reichert (1986 Part I), Dehez & Drèze (1988 b), Mas-Colell & Silvestre (1989) and Hart & Mas-Colell (1990).

5.- THE EFFICIENCY PROBLEM

It has already been mentioned that the interest of Marginal Pricing derives from the fact that it satisfies the necessary conditions for optimality. It is time now to be more precise about this, and to address the question concerning sufficiency. It will be shown first that, under very general assumptions, any Pareto optimal allocation can be decentralized as a Marginal Pricing Equilibrium. This amounts to saying that Marginal Pricing is a necessary condition in order to achieve efficiency through a price mechanism. Yet Marginal Pricing is far from being sufficient, as it will be illustrated by a number of examples. Hence we are facing almost an impossibility result: Under general conditions, there is no way of allocating efficiently the resources through a price mechanism, in the presence of increasing returns to scale. A more general question arises then: the analysis of the nonemptyness of the core in an economy with increasing returns.

5.1.- The Second Welfare Theorem with Increasing Returns.

Let $E = \{ (X_i, u_i), (Y_j), \omega \}$ describe our economy of reference, that is, an economy with ℓ commodities, m consumers (characterized by their consumption sets and utility functions, X_i, u_i, respectively), n firms (characterized by their production sets Y_j), and a vector of initial endowments $\omega \in \mathbb{R}^{\ell}$. Consider now the following assumption (which is a weakening of assumptions (A.1) and (A.3) in Section 2):

H.- (i) For every $j = 1, 2, \ldots, n$, Y_j is a closed subset of $\mathbb{R}^\ell$ such that $Y_j - \mathbb{R}^\ell_+ \subset Y_j$.

(ii) For every $i = 1, 2, \ldots, m$, $X_i \subset \mathbb{R}^\ell$ is closed and convex, and $u_i : X_i \longrightarrow \mathbb{R}$ is a continuous and quasi-concave function, which satisfies local non-satiation.

The following result [due originally to Guesnerie (1975)] is an extension of the Second Welfare Theorem to economies with nonconvex production sets [see Vohra (1991, Th. 1)]:

Theorem 5.1.- Let $E = \{ (X_i, u_i), (Y_j), \omega \}$ be an economy satisfying assumption (H), and let $[(x^*_i), (y^*_j)]$ be a Pareto Optimal allocation. Then, there exists $p \in \mathbb{R}^\ell_+$, $p \neq 0$, such that:

(a) For all i, $u_i(x_i) \geq u_i(x^*_i) \Longrightarrow px_i \geq px^*_i$.

(b) $p \in \phi^{MP}_j(p, \bar{y}^*)$, $\forall j$.

Remark 5.1.- Let $[(x^*_i), (y^*_j)]$ be a Pareto Optimal allocation, and suppose that for some consumer u_i is differentiable at $x^*_i \in \text{int}X_i$. Then, for this consumer, the (normalized) vector of marginal rates of substitution is unique. Thus the price vector supporting that allocation turns out to be unique.

Theorem 5.1 provides an extension of the Second Welfare Theorem allowing for nonconvex production sets. It tells us that any efficient allocation can be decentralized as a marginal pricing equilibrium, provided we are free to carry out any feasible lump-sum transfer which may be required. This suggests that the way of interpreting marginal rates of transformation as Clarke's normal cones is appropriate. The remark above reinforces such an idea: it says that (under very mild regularity conditions) marginal pricing is a necessary condition for achieving Pareto Optimality through a price mechanism.

Thus, in the context of a regulated economy where arbitrary lump-sum transfers are possible, efficiency can be obtained by instructing firms to follow marginal pricing. Notice that when production sets are convex, marginal pricing corresponds to profit maximization. Therefore we can interpret this result in terms of a mixed economy with a competitive sector (convex firms) and a regulated one, where all firms follow marginal pricing, and efficiency is obtained by suitably redistributing wealth.

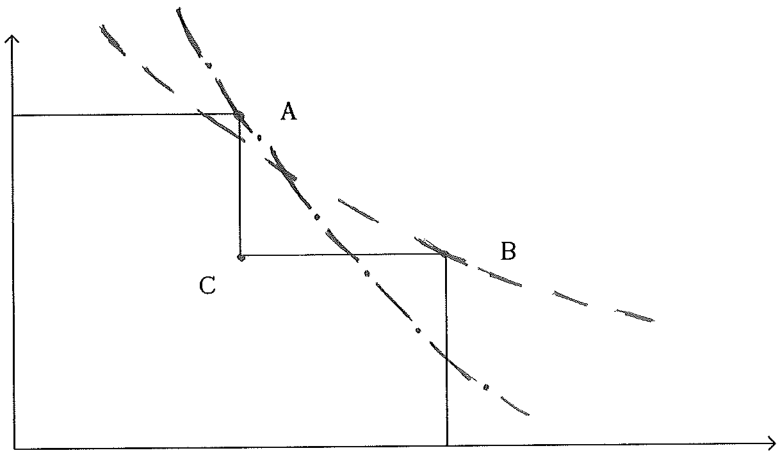
5.2.- The Failure of the First Welfare Theorem.

It is not true, however, that Marginal Pricing implies optimality, that is, Marginal Pricing is a necessary but not a sufficient condition for optimality (a general problem for nonconvex programming). To see this we shall briefly report on three key examples. In the first one, no marginal pricing equilibrium is Pareto efficient [the example is developed in Brown & Heal (1979), after Guesnerie's (1975) previous one]. In the second one [due to Beato & Mas-Colell (1983), (1985)], it is shown that even production efficiency may fail in a Marginal Pricing Equilibrium. Finally, the third example [Vohra (1988 b)] presents a situation where

Marginal Pricing is Pareto dominated by Average Cost Pricing (and thus is not even second best efficient). Each of these examples illustrates different aspects of the problem.

<u>Example</u> <u>1.-</u> [see Brown & Heal (1979) for details]

Consider an economy with two goods, a single non-convex firm and two consumers. Figure 5.1 illustrates the three possible equilibria, which correspond to points A, B and C (we take the two commodities to be measured by nonnegative numbers). The dashed lines S_A, S_B describe the Scitovsky's community indifference curves for A and B (these curves describe the boundary of the aggregate "better than" sets at points A and B). Since both curves are above point C, this is clearly an inefficient production plan. But marginal prices supporting S_A at A, cut the production set below B, so that A cannot correspond to an efficient allocation. Similarly, marginal prices supporting S_B at B cut the production set below A, which means that B cannot be efficient either. Hence, none of the marginal pricing equilibria is efficient.



- Figure 5.1 -

Example 2.- [see Beato & Mas-Colell (1983), (1985)]

Consider an economy with two goods, $h = 1, 2$. Good 1 is used as an input to produce Good 2. There are two firms, $j = 1, 2$ whose production possibilities are described as follows (note that the first one exhibits constant returns to scale, while the second one has increasing returns):

$$Y_1 \equiv \{ \mathbf{y}_1 \in \mathbb{R}^2_+ \ / \ (y_{11}, y_{12}) \leq (y_{11}, y_{11}) \}$$

$$Y_2 \equiv \{ \mathbf{y}_2 \in \mathbb{R}^2_+ \ / \ (y_{21}, y_{22}) \leq [\ y_{21}, \ \frac{1}{16}(y_{21})^2 \ ] \}$$

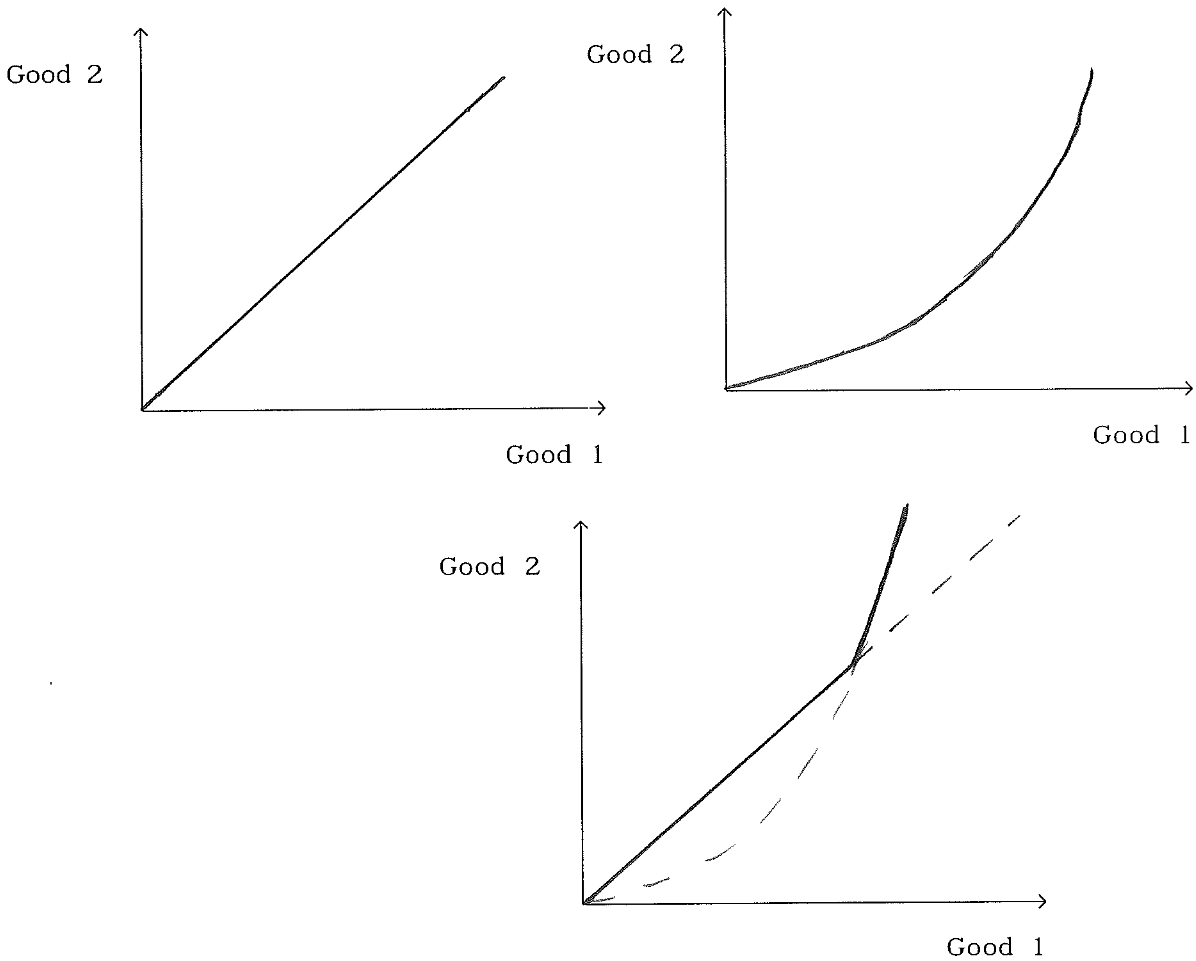
Figure 5.2 shows the individual and aggregate technologies.

There are two consumers whose characteristics are described by:

$$X_1 = \mathbb{R}^2_+ \ , \qquad u_1(\mathbf{x}_1) = x_{12} \ , \qquad \omega_1 = (0, 50)$$

$$X_2 = \mathbb{R}^2_+ \ , \qquad u_2(\mathbf{x}_2) = \text{Min.} \{ 6x_{21}, x_{22} \} \ , \qquad \omega_2 = (20, 0)$$

From this it follows that we can take $p_2 = 1$ (since commodity 2 will always have a positive price in equilibrium, in view of u_1). Then, given the first firm's production set, an equilibrium can only occur for $p_1 \geq 1$ (and positive production for the first firm actually requires $p_1 = 1$). For $p_1 = 1$, the production equalizing the second firm's marginal rate of transformation (and actually yielding a Marginal Pricing equilibrium) implies an inefficient aggregate production plan, since in this economy efficiency implies that only one of the firms must be active in equilibrium.



- Figure 5.2 -

Example 3.- [Vohra (1988 b)]

Let a private ownership economy with two goods, two consumers and a single non-convex firm, whose data are summarized as follows:

$$X_1 = \mathbb{R}^2_+ , \quad u_1(\mathbf{x}_1) = x_{12} , \quad \omega_1 = (0, 10) , \quad \theta_1 = 1$$

$$X_2 = \mathbb{R}^2_+ , \quad u_2(\mathbf{x}_2) = 4 \log. x_{21} + x_{22} , \quad \omega_2 = (20, 0) , \quad \theta_2 = 0$$

$Y = \{ \mathbf{y} \in \mathbb{R}^2 \;/\; y_1 \leq 0;\ y_1 + y_2 \leq 0$ if $y_1 \geq -16$ and $10y_1 + y_2 + 144 \leq 0$ if $y_1 \leq -16 \}$

In view of u_1 we can take $p_2 = 1$. It can be shown that the only Marginal Pricing equilibrium of this economy corresponds to:

$$\mathbf{p}^* = (1, 1), \quad \mathbf{y}^* = (-16, 16), \quad \mathbf{x}^*_1 = (0, 10), \quad \mathbf{x}^*_2 = (4, 16)$$

Let us think now of the situation corresponding to an Average Cost Pricing equilibrium. It can be checked that

$$\mathbf{p}' = (2, 1), \quad \mathbf{y}' = (-18, 36), \quad \mathbf{x}'_1 = (0, 10), \quad \mathbf{x}'_2 = (2, 36)$$

is an Average Cost Pricing equilibrium. Notice that the first consumer's utility is the same as in the Marginal Pricing Equilibrium ($u'_1 = 10$), while the second consumer is now better-off (since $u'_2 = 4 \log. 2 + 36$ is greater than $u^*_2 = 4 \log. 4 + 16$).

This asymmetry between the validity of the Second Welfare Theorem and the failure of the First one points out that, for some economies, there are rules of income distribution which may be inherently incompatible with efficiency. The reason is that, contrary to the convex case, the mapping associating efficient allocations to income distributions is not onto. Thus, for fixed income distribution schemes, we can find non-convex economies such that the agents' characteristics (technology and preferences) are such that marginal pricing generates an income distribution which has an empty intersection with the subset of efficient income distributions. The three examples show this feature. The second one also indicates that Marginal Pricing equilibria can be associated with a number of active firms which is inadequate (up to a point where even aggregate production efficiency is violated). The third example tells us that there may be better alternatives than Marginal Pricing in specific contexts where Pareto optimality fails.

Theorem 5.1 shows that if any feasible income redistribution is possible, one can obtain Pareto efficient allocations as Marginal Pricing equilibria; on the other hand, the examples illustrate that if the income distribution rule is fixed, then in general Marginal Pricing does not imply optimality. A natural question is then whether there is some possibility of obtaining efficiency in the case of fixed distribution rules which are supplemented by some limited transfers.

Vohra's (1991) paper addresses this point, by considering the case where transfers can only be used in order to finance the possible losses of nonconvex firms, and not for redistribution purposes. Thus these transfers will be taxes if nonconvex firms have losses, and subsidies otherwise, so that "no consumer is subsidized if some other consumer is taxed". To be precise, let us think of a mixed economy where firms 1, 2, ..., h are competitive (i.e., they behave as profit maximizers at given prices, over convex sets), while firms $h+1$, $h+2$, ..., n are non-convex. Let us define then a Tax Structure as a system of transfers $(t_i) \in \mathbb{R}^m$ such that:

$$\sum_{i=1}^{m} t_i = - \sum_{j=h+1}^{n} p\, y_j, \quad \text{and either } t_i \geq 0 \;\; \forall\, i, \text{ or } t_i \leq 0 \;\; \forall\, i$$

The *ith* consumer budget set is then given by:

$$\beta_i(p, \bar{y}, t_i) \equiv \{ x_i \in X_i \;/\; px_i \leq p\omega_i + \sum_{j=1}^{h} py_j - t_i \}$$

Vohra establishes then the following definition:

Definition 5.1.- A **Regulated Market Equilibrium** is a point

$$[p^*, (x_i^*), (y_j^*), (t_i^*)] \in \mathbb{P} \times \prod_{i=1}^{m} X_i \times \prod_{j=1}^{n} Y_j \times \mathbb{R}^m$$

such that:

(i) For every i, $u_i(x_i^*) \geq u_i(x_i)$, $\forall \; x_i \in \beta_i(p^*, \bar{y}^*, t_i^*)$

(ii) For $j = 1, 2, \ldots, h$, $p^*y_j^* \geq p^*y_j \;\; \forall\, y_j \in Y_j$

(iii) (t_i^*) is a Tax Structure

(iv) $\sum_{i=1}^{m} x_i^* = \sum_{j=1}^{n} y_j^* + \omega$

Observe that the only restriction that this definition establishes over non-convex firms is that the equilibrium production plans must be technologically feasible. This allows for different types of behaviour of non-convex firms. In particular, given a private ownership economy E, a

Marginal Cost Pricing Equilibrium with a Tax Structure (t_i^o) can be defined as a Regulated Market Equilibrium [p^*, (x_i^*), (y_j^*), (t_i^o)], where $p^* \in \bigcap_{j=h+1}^{n} \phi_j^{MP}(p^*, \bar{y}^*)$. The examples above show that there are economies such that, for a given Tax Structure (t_i^o), none of the Marginal Pricing Equilibria with a Tax Structure (t_i^o) satisfies Pareto Optimality, Production efficiency or Second Best Efficiency.

If this happens for a given tax structure, the question is then whether we can find appropriate tax structures, depending on each specific economy, to circumvent these negatives results. Vohra (1991, Sec. 3 & 4) shows that, in this general context, there is little hope of finding optimal allocations via Marginal Pricing and a suitable choice of a Tax Structure. He provides an example which proves the following assertion [see Vohra (1991, Prop. 4)]:

Proposition 5.1.- There exists a class of economies in which there does not exist any Pareto optimal Regulated Market Equilibrium. In particular, there is no Tax Structure and corresponding to it a Marginal Pricing Equilibrium which is Pareto optimal.

Thus this result tells us that there is no general way of ensuring Pareto optimality by instructing non-convex firms to follow Marginal Pricing, if we are not ready to perform an explicit redistribution policy.

It is an immediate consequence of Theorem 5.1 and the definitions above that, for any given Tax Structure (t_i), a Pareto optimal allocation can be decentralized as a Marginal Pricing Equilibrium with a Tax Structure (t_i). Then, by noticing that Two-part Marginal Tariffs are equivalent to marginal pricing combined with a particular Tax Structure (the entrance fee), the following conclusions obtain:

(i) Any efficient allocation can be decentralized as a Two-part Marginal Tariff equilibrium, provided there is sufficient willingness to pay [see Quinzii (1991) and Brown, Heller & Starr (1992)].

(ii) The partial equilibrium intuition about obtaining efficiency through nonlinear prices does not hold in a general equilibrium framework [see Vohra (1990)].

5.3.- The Core of an Economy with Increasing Returns.

Marginal Pricing is practically a necessary condition for Pareto optimality, when we come to allocate the resources through a price mechanism. Yet it has been shown to be highly insufficient (Proposition 5.1 showed that there is no general way of ensuring Pareto optimality by instructing non-convex firms to follow Marginal Pricing, if we are not ready to perform an explicit redistribution policy). This impossibility result suggests dealing with the problem from a more general perspective, that is, analyzing the compatibility of increasing returns and efficiency without requiring the existence of a price mechanism. More precisely, it raises the question of the existence of core allocations (namely, allocations such that no coalition can improve upon by using their own endowments and the available technology).

At first glance, one would expect that the presence of increasing returns may facilitate the nonemptyness of the core: bigger coalitions are more likely to get higher productivity. This intuition, however, is far from reality, as shown in Scarf (1986) (a paper written in 1963).

We shall specialize our reference model, in order to address this problem. The following assumptions incorporate two main restrictions: (i) There is a single production set; and (ii) Consumption plans are represented by nonnegative vectors. Formally:

A.1'.- There is a single production set Y, which is a closed subset of $\mathbb{R}^{\ell}$, such that $0 \in Y$, and $Y - \mathbb{R}^{\ell}_{+} \subset Y$.

A.2'.- For each given vector of initial endowments $\omega \in \mathbb{R}^{\ell}$, and every $(x_i) \in \prod_{i=1}^{m} X_i$ the set of attainable productions is compact.

A.3'.- For each $i = 1, 2,..., m$: (a) $X_i = \mathbb{R}^{\ell}_{+}$; (b) $u_i : X_i \longrightarrow \mathbb{R}$ is a continuous and quasi-concave function, which satisfies Local Non-Satiation; and (c) $\omega_i >> 0$.

The following result tells us the bad news [see Scarf (1986), Quinzii (1992, Th. 6.2)]:

Theorem 5.2.- Let $\{ (X_i, u_i, \omega_i), Y \}$ represent the set of economies satisfying assumptions (A.1'), (A.2') and (A.3'). The core of every such economy is non-empty if and only if Y is a convex cone.

This result says that with the degree of generality given by assumptions (A.1'), (A.2') and (A.3') we can always construct economies with increasing returns and an empty core. Hence, the difficulties between increasing returns and efficiency are somehow more substantial than the way of pricing commodities.

In spite of this discouraging result, Scarf (1986) also identifies a particular family of economies satisfying assumptions (A.1'), (A.2') and (A.3') with nonempty cores. For that he introduces the notion of a Social Equilibrium. A Social Equilibrium consists of a price vector and a feasible allocation such that consumers satisfy their preferences, the firm maximizes profits within the set of feasible productions (i.e., those using no more inputs than those available), and equilibrium profits are null. There are two special assumptions in Scarf's (1986) model which allow him to ensure that a Social Equilibrium exists and it is in the core. The first one is the distinction between "two types of commodities: consumer goods, which appear in consumers' utility functions, and producer goods or inputs to production, which do not" [Cf. Scarf (1986, p. 403)]. The second one consists of assuming that the production set is input-distributive.

In order to be precise about these points, let us introduce the following assumption:

A.0'.- The set $\mathcal{L} \equiv \{ 1, 2, \ldots, \ell \}$ can be partitioned into two disjoint subsets, *PC* and its complement, so that if $\mathbf{y} \in Y$ and $t \in PC$, then $y_t \leq 0$. We shall refer to goods in *PC* as Producer Commodities, and write production plans as $\mathbf{y} = (\mathbf{a}, \mathbf{b})$, with $\mathbf{a} \leq 0$, in the understanding that $\mathbf{a}$ is a point in the subspace of Producer Commodities.

Assumption (A.0') makes a distinction between two groups of commodities: Producer Commodities (which are negative), and other commodities. Note that there is no sign restriction over $\mathcal{L}\backslash PC$, so that it may well be factors of production in this set (e.g. labour).

Under assumption (A.0'), consider a market economy as in Section 3. The following definition makes precise the notion of Social Equilibrium:

<u>Definition</u> <u>5.2.-</u> We shall say that a price vector, $\mathbf{p}^* \in \mathbb{P}$, and an allocation, $[(\mathbf{x}_i^*), \mathbf{y}^*]$, yield a **Social Equilibrium** if the following conditions are satisfied:

(α) For each $i = 1, 2, \dots, m$, $\mathbf{x}_i^* \in \xi_i(\mathbf{p}^*, \bar{\mathbf{y}}^*)$

(β) $0 = \mathbf{p}^*\mathbf{y}^* \geq \mathbf{p}^*\mathbf{y}$, $\forall\ \mathbf{y} \in Y$ such that $\mathbf{a} \geq \mathbf{a}^*$

(γ) $\sum_{i=1}^{m} \mathbf{x}_i^* - \mathbf{y}^* = \omega$

That is, a Social Equilibrium is a situation in which: (a) Consumers maximize their preferences subject to their budget constraints; (b,1) The unique firm maximizes profits at given prices, subject to an input constraint; (b,2) Equilibrium profits are null; and (c) All markets clear.

We shall give now the definition of input distributive sets, assuming that (A.0') holds. As before, for any finite collection $(\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^k)$ of production plans, $\mathbf{y}^t \in Y$ for all t, we shall denote by $K(\mathbf{y}^1, \dots, \mathbf{y}^k)$ the convex cone with vertex zero generated by points $(\mathbf{y}^1, \dots, \mathbf{y}^k)$.

Definition 5.3.- We shall say that Y is an **Input-Distributive Set** whenever, for any finite collection $(y^1, y^2, \ldots, y^k)$ of production plans, with $y^t = (a^t, b^t)$ and $y^t \in Y$ for all t, the following inclusion is satisfied:

$$K(y^1, \ldots, y^k) \cap \{ y \in \mathbb{R}^\ell \ / \ y = (a, b) \ \& \ a \leq a^t \ \forall t \} \subset Y$$

Thus a production set is said to be Input-Distributive if any (nonnegative) weighted sum of feasible production plans is feasible if it does not use fewer Producer Commodities than any of the original plans. It can be checked that input-distributivity implies non-decreasing returns to scale, and that the iso-inputs sets,

$$B(a) = \{ b \ / \ (a, b) \in Y \}$$

are non-empty, closed and convex sets.

The following theorem gives us the desired result [see Scarf (1986)]:

Theorem 5.3.- Let E be an economy satisfying assumptions (A.0'), (A.1'). (A.2') and (A.3'). Suppose furthermore that Y is an input-distributive set, and that Producer Commodities are not consumed. Then a Social Equilibrium exists and it is in the core.

Unfortunately, this result does not extend to the case of several nonconvex firms, since Input-Distributivity is a property which is not preserved by summation.

References to the literature.- In a remarkable paper, Guesnerie (1975) showed that marginal pricing is a necessary condition for optimality. He did that by extending the Second Welfare Theorem to economies with non-convex production sets, and using the Dubovickii-Miljutin cones of interior displacements as the main tool to extend the notion of marginal pricing to non-smooth, non-convex sets. After Cornet's introduction of the more general Clarke's normal cones for this type of analysis, Khan & Vohra (1987) extended this result to economies with public goods, and Bonnisseau & Cornet (1988 b) to economies with an infinite dimensional commodity space [see also Cornet (1986)]. Vohra (1991) and Quinzii (1992, Ch. 2) provide elegant and easy proofs of this result.

The failure of Marginal Pricing equilibria to achieve Pareto optimality was also shown in Guesnerie (1975) (he gave the first example of an economy where <u>all</u> marginal pricing equilibria were inefficient). Additional examples of this phenomenon appeared in Brown & Heal (1979). Beato & Mas-Colell (1983) provided a first example in which marginal pricing equilibria were not in the set of efficient aggregate productions. Vohra (1988 b) develops a systematic analysis of the inefficiency of marginal pricing for fixed rules of income distribution. Vohra (1990) shows that the partial analysis intuition about the possibility of obtaining Pareto efficient allocations via two-part marginal pricing, does not work. An excellent exposition of the efficiency problems in this context appears in Vohra (1991). Brown, Heller & Starr (1992) prove that efficient allocations can be decentralized as Two-Part Marginal Equilibria, provided there is sufficient willingness to pay.

Some positive results are available for the case of a single non-convex firm. Brown & Heal (1983) showed that assuming homothetic preferences (which implies that Scitovsky's community indifference curves do not intersect), there exists at least a Pareto optimal marginal pricing equilibrium. Sufficient conditions for the optimality of marginal pricing in a more general context are analyzed in Dierker (1986) and Quinzii (1991). These conditions refer to the relative curvature of the production frontier and of the community indifference curves, so that when the social indifference curve is tangent to the feasible set it never cuts inside it. See also the the special cases analyzed in Vohra (1990), (1991).

Concerning the two welfare theorems in economies with increasing returns, see the illuminating discussions in Guesnerie (1990), Vohra (1991) and Quinzii (1992 Chs. 1 - 4).

An excellent and very detailed discussion of the nonemptyness of the core of an economy with nonconvex technologies can be found in Quinzii (1992, Ch. 6). Besides Scarf's (1986) paper, it is worth mentioning the contributions of Quinzii (1982), Ichiischi & Quinzii (1983) (dispensing with the requirement of "inputs which are not consumed"), and Reichert (1986) (who uses a nonlinear single-production input-output model, to allow for the presence of many nonconvex firms).

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