

**FAIR ALLOCATION IN A GENERAL MODEL
WITH INDIVISIBLE GOODS***

Carmen Beviá**

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A B S T R A C T

In this paper we study the problem of fair allocation in economies with indivisible goods, dropping the usual restriction that one agent receives at most one indivisible good. We show that most of the results obtained in the literature do not hold when the above mentioned restriction is dropped.

KEYWORDS: Indivisible Goods, Consistency, Population Monotonicity.

1. INTRODUCTION

The problem of fair allocation in economies with indivisible goods has been studied under the assumption that one agent receives at most one indivisible good. These economies will be called *traditional economies*.

There are many situations in which the restriction is totally meaningful, for example those cases in which the indivisible goods are jobs or grants. However, in some other situations, we would like to allow some agents to receive more than one good; think of the problem of allocating an inheritance among a group of heirs. If, for one of these heirs, a certain group of objects has special sentimental value, why can he not obtain more than one object if he is able, in turn, to compensate the rest of the heirs?.

The aim of our study is to find out to what extent the results obtained in the existing literature depend on that restriction. For this purpose, we present a general model where each agent can receive more than one indivisible good and preferences are quasi-linear.

There are two kinds of results in the literature. Some pertain to the relationship among concepts such as *Pareto* solution; *No-Envy* solution; *Group-No-Envy* solution; *Equal Income Walrasian* solution; *Equal Split Guarantee* solution. On the other hand, the desirability of some of these concepts have been studied in the light of certain properties like *consistency*, and

population-monotonicity. In this paper we will show that most of the results obtained in the framework of *traditional economies* do not, in general, hold.

In **traditional economies**, some of the relations among the above mentioned solutions are,

Proposition 1.1. (Svensson (1983)). The *No-Envy* solution is a subsolution of the *Pareto* solution.

Proposition 1.2. (Svensson (1983)). The *Group-No-Envy* solution and the *Equal Income Walrasian* solution coincide with the *No-Envy* solution.

Proposition 1.3. (Bevia (1993)). The *No-Envy* solution is a subsolution of the *Equal Split Guarantee* solution. Moreover, these solutions coincide for economies with two agents.

None of these results hold in the general model.

Next we turn our attention to those properties that are considered as desirable for a solution, for example, *consistency*. In traditional economies, this property does not allow us to make any selection from the set of *envy-free* allocations. That is, if *consistency* is required, the whole set of *envy-free* allocations should be accepted.

Proposition 1.4. (Tadenuma and Thomson (1991)). There is no proper subsolution of the *No-Envy* solution that satisfies *consistency* and *neutrality*.

In the general model this result does not hold, since the *Group-No-Envy* solution and the *Equal Income Walrasian* solution are proper subsolutions of the *No-Envy* solution which satisfy *consistency* and *neutrality*; interestingly, they are not the only ones. Another example of such a subsolution is presented in the paper.

In traditional economies, a positive result is obtained if *consistency* is replaced by *bilateral consistency*. However, if *converse consistency* is added, an impossibility is, one again, obtained.

Proposition 1.5. (Tadenuma and Thomson (1991)). There are proper subsolutions of the *No-Envy* solution satisfying *bilateral consistency* and *neutrality*. All such subsolutions coincide with the *No-Envy* solution in economies with two agents.

Proposition 1.6. (Tadenuma and Thomson (1991)). There is no proper subsolution of the *No-Envy* solution satisfying *bilateral consistency*, *converse consistency* and *neutrality*.

Although these results do not hold in our model (actually, the *No-Envy* and *Pareto* solution does not satisfy *converse consistency*), we obtain generalizations by adding a new property that we call *Property B*.

In the last part of the paper, we study the *population-monotonicity* property. Alkan (1989) showed that there is no selection from the *No-Envy* solution which satisfies it. This impossibility remains true in our model. In

fact, we obtain a stronger impossibility result, namely, *there is no subsolution from the Pareto solution satisfying population monotonicity*. However, in traditional economies, selections from the *Pareto* solution satisfying *population-monotonicity* exist, as stated next.

Proposition 1.7. (Moulin (1992)). The *Shapley* solution is a *Pareto* solution which satisfies *population-monotonicity*.

In our model, the above result is only true if we impose a certain kind of *substitutability* condition. In traditional economies this condition is a consequence of the fact that each agent receives at most one indivisible good.

The rest of the paper is organized as follows. In Section 2 we present the model, the main definitions and the relationships among the solutions under consideration. In Section 3 we study the problem of fair allocation from the point of view of *consistency*. In Section 4 we present the results related to *population-monotonicity*. In order to keep the reading easy, some proofs are relegated to the Appendix.

2. THE MODEL.

An **economy** is a list $e = (Q, \Omega, M; R_Q)$, where Q is a finite set of agents with elements i, j, k, \dots , Ω is a finite set of objects with elements $\alpha, \beta, \gamma, \dots$, $M \in \mathbb{R}$ is an amount of money, and $R_Q = (R_i)_{i \in Q}$ is a list of preference relations defined over $\mathcal{P}(\Omega) \times \mathbb{R}$, where $\mathcal{P}(\Omega)$ denotes the power set of Ω , that is, the set of all subsets of Ω .

Let P_i denote the strict preference relation associated to R_i , and I_i the indifference relation. Each preference relation is assumed to be continuous and increasing in money. Thus,

$$\text{if } m > m', \text{ then } (\mathcal{A}, m) P_i (\mathcal{A}, m') \text{ for all } i \in Q \text{ and for all } \mathcal{A} \in \mathcal{P}(\Omega) \quad (1)$$

Also it is assumed that for any pair $(\mathcal{A}, m) \in \mathcal{P}(\Omega) \times \mathbb{R}$, for any $i \in Q$ and for any $\mathcal{B} \in \mathcal{P}(\Omega)$, an amount of money m' exists such that $(\mathcal{A}, m) I_i (\mathcal{B}, m')$ (2)

Let \mathcal{E} be the class of all economies fulfilling (1) and (2) above.

Given an economy $e \in \mathcal{E}$, let $q = |Q|$, and let $\mathcal{P}(\Omega, q)$ be the set of all the partitions of Ω with q elements, that is,

$$\mathcal{P}(\Omega, q) = \{ \mathcal{P} = \{\mathcal{A}_1, \dots, \mathcal{A}_q\} / \forall i \mathcal{A}_i \in \mathcal{P}(\Omega); \bigcup \mathcal{A}_i = \Omega; \mathcal{A}_i \cap \mathcal{A}_j = \emptyset; \forall i \neq j \}.$$

Notice that an element in $\mathcal{P}(\Omega, q)$ can be such that $\mathcal{A}_i = \{\Omega\}$ for some $i \in Q$, and $\mathcal{A}_j = \{\emptyset\}$ for all $j \neq i$.

A feasible allocation for e is a pair $z = (\sigma, m)$, where $\sigma: Q \rightarrow \mathcal{P}$ is a bijection, $\mathcal{P} \in \mathcal{P}(\Omega, q)$, and $m = (m_{\sigma(i)})_{i \in Q} \in \mathbb{R}^q$ is such that $\sum_{i \in Q} m_{\sigma(i)} = M$. The bijection σ is called an assignment. For each $i \in Q$, let $z_i = (\sigma(i), m_{\sigma(i)})$ be the consumption of agent i . Note that $m_{\sigma(i)}$ can be positive, negative or zero. Let $Z(e)$ be the set of feasible allocations for e .

A solution on \mathcal{E} associates a non-empty subset of $Z(e)$ to every $e \in \mathcal{E}$. Given two solutions, φ and ψ , $\varphi\psi$ denotes their intersection, whenever it is not empty. The definition below is a primary example of a solution:

The Pareto solution, P: given $e \in \mathcal{E}$,

$$P(e) = \{z \in Z(e) / \exists z' \in Z(e) \text{ s.t. } \forall i \in Q, z'_i R_i z_i \text{ and } \exists j \in Q \text{ s.t. } z'_j P_j z_j\}$$

An economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ is quasi-linear if each $i \in Q$ has preferences such that for all $\mathcal{A} \in \mathcal{P}(\Omega)$ and for all $m, m', t \in \mathbb{R}$, $(\mathcal{A}, m) I_i (\mathcal{B}, m')$ implies $(\mathcal{A}, m+t) I_i (\mathcal{B}, m'+t)$. Let $\mathcal{E}_{ql} \subseteq \mathcal{E}$ be the subclass of quasi-linear economies. The rest of the paper is developed in the context of quasi-linear economies.

Given an economy $e \in \mathcal{E}_{ql}$, for each $i \in Q$ and for each $\mathcal{A} \in \mathcal{P}(\Omega)$, let $v_{i\mathcal{A}} \in \mathbb{R}$ be such that, $(\mathcal{A}, 0) I_i (\emptyset, v_{i\mathcal{A}})$. Each $v_{i\mathcal{A}}$ can be interpreted as the valuation of agent i of the set of objects \mathcal{A} . By definition $v_{i\emptyset} = 0$ for all $i \in Q$. From (1) and (2), $v_{i\mathcal{A}}$ exists and is unique. Notice that in quasi-linear economies preferences are completely described by these numbers. In the rest of the paper we will use the following identification, for each $i \in Q$,

$$R_i \equiv (v_{i\mathcal{A}})_{\mathcal{A} \in \mathcal{P}(\Omega)}$$

In \mathcal{E}_{q1} the set of *Pareto efficient* allocations can be characterized as stated in Lemma 2.1.

Lemma 2.1. Given $e \in \mathcal{E}_{q1}$, an allocation $(\sigma, m) \in Z(e)$ is *Pareto efficient* for e if and only if for any other assignment τ ,

$$\sum_{i \in Q} v_i \sigma(i) \geq \sum_{i \in Q} v_i \tau(i)$$

Proof. See Appendix.

This result is not surprising at all, in fact, it is a generalization of that obtained in traditional economies.

The above characterization allows us to give the following definition,

An efficient assignment for e is an assignment σ from Q to \mathcal{P} such that for any $\mathcal{P}' \in \mathcal{P}(\Omega, q)$, and any assignment τ from Q to \mathcal{P}' ,

$$\sum_{i \in Q} v_i \sigma(i) \geq \sum_{i \in Q} v_i \tau(i)$$

Let $\Sigma(e) = \{\sigma / \sigma \text{ is an efficient assignment for } e\}$.

One of the most studied solutions in the literature on fair allocation is the *No-Envy* solution. This solution selects the allocations at which no agent prefers any other agent's bundle to his own.

The No-Envy solution, N , (Foley, 1967): given $e \in \mathcal{E}$,

$$N(e) = \{ z \in Z(e) / \forall i, j \in Q, z_i R_i z_j \}$$

If $z \in N(e)$, z is called an **envy-free** allocation.

The first difference between our model and the traditional one is that not all the *envy-free* allocations are *Pareto efficient*. It should be pointed out that in economies with divisible goods the same result is obtained.

Proposition 2.1. An *envy-free* allocation is not necessarily *Pareto efficient*.

Proof. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$, $v_{1\alpha} = 3$, $v_{1\beta} = 1$, $v_{1(\alpha, \beta)} = 8$, and $v_{2\alpha} = 1$, $v_{2\beta} = 2$, $v_{2(\alpha, \beta)} = 4$. Let $z = ((\{\alpha\}, -1), (\{\beta\}, 1)) \in Z(e)$. It is easy to check that z is *envy-free*, but it is not Pareto efficient, because from Lemma 2.1 any *Pareto efficient* allocation is such that $\sigma(1) = \{\alpha, \beta\}$ and $\sigma(2) = \{\emptyset\}$. ■

In the class of quasi-linear economies it is easy to prove the existence of *envy-free* and *Pareto efficient* allocations. The argument that we use to prove it is the same as that used in Alkan-Demange-Gale (1991) to prove the existence of *envy-free* allocations in traditional economies with quasi-linear preferences. Notice that if σ is an efficient assignment and z is an *envy-free* allocation then z is *efficient*.

Proposition 2.2. For all $e \in \mathcal{E}_{q1}$, $NP(e)$ is not empty.

Proof. See Appendix.

In the existence proof of the above proposition it was essential that the money vector m could have positive or negative entries. In the usual fair allocation problem one requires that all quantities be non-negative. If we insist on non-negative quantities of money, we should guarantee that there is enough money to compensate the agents who do not get the objects they most prefer. The next proposition identifies this amount of money.

Proposition 2.3. Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, if $M \geq qv_{iA} - \sum_{B \in \mathcal{P}} v_{iB}$ for all $i \in Q$, for all $A, B \in \mathcal{P}$, for all $\mathcal{P} \in \mathcal{P}(\Omega, q)$, then there exist $z = (\sigma, m) \in N(e)$ such that, $m_{\sigma(i)} \geq 0$ for all $i \in Q$. Moreover, for all $z = (\sigma, m) \in N(e)$, $m_{\sigma(i)} \geq 0$ for all $i \in Q$.

Proof. See Appendix

Other solutions have been studied in relation with the problem of fair allocation. For example, the *Group-No-Envy* solution. This solution selects the allocations at which no group is able to make all of its members at least as well off, and one of them strictly better off, if they have access to the resources attributed to any other group of the same size.

The Group-No-Envy solution, G , (Vind (1971), Varian (1974)): given $e \in \mathcal{E}$, $z = (\sigma, m) \in G(e)$ if $z \in Z(e)$ and for all groups $C, C' \subseteq Q$ with $|C| = |C'|$,

there is no $z' = (\tau, m') \in Z(e)$ such that $\sum_{i \in C} m'_{\tau(i)} = \sum_{j \in C'} m_{\sigma(j)}$, $U_{i \in C} \tau(i) = U_{j \in C'} \sigma(j)$, $z'_1 R_1 z_1$ for all $i \in C$, and $z'_j P_j z_j$ for at least one $j \in C$.

If $z \in G(e)$, z is called a **group-envy-free** allocation.

Note that any *group-envy-free* allocation is both *Pareto efficient* (take $C=C'=Q$), and *envy-free* (take $|C|=|C'|=1$).

In our model, this solution is more restrictive than the *No-Envy* solution in contrast with the traditional model.

Proposition 2.4. The *Group-No-Envy* solution is a proper subsolution of the *No-Envy* and *Pareto* solution.

Proof. We know that the *Group-No-Envy* solution is a subsolution of the *No-Envy* solution. In order to prove that it is a proper subsolution we give an example of an economy where there are allocations that are *envy-free* and *Pareto efficient* and are not *group-envy-free*.

Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_s$ be such that $Q = \{1, 2, 3\}$, $\Omega = \{\alpha, \beta, \gamma\}$, $M = 0$, $v_{1\alpha} = 3$, $v_{1\beta} = 1$, $v_{1\gamma} = 1$; $v_{2\alpha} = 3$, $v_{2\beta} = 3$, $v_{2\gamma} = 1$; $v_{3\alpha} = 1$, $v_{3\beta} = 0$, $v_{3\gamma} = 1$, and for any $A \in \mathcal{P}(\Omega)$, for any $i \in Q$, $v_{iA} = \sum_{\delta \in A} v_{i\delta}$. Let $z = (\sigma, m) = ((\{\alpha\}, -1), (\{\beta\}, 1), (\{\gamma\}, 0)) \in Z(e)$. Clearly, z is *Pareto efficient* and *envy-free*. Let $C = \{2, 3\}$ and $C' = \{1, 2\}$, and let $z' = (\tau, m') = ((\{\gamma\}, 0), (\{\alpha, \beta\}, -1.5), (\{\emptyset\}, 1.5)) \in Z(e)$. This z' satisfies, $\sum_{i \in C} m'_{\tau(i)} = 0 = \sum_{j \in C'} m_{\sigma(j)}$, $U_{i \in C} \tau(i) = \{\alpha, \beta\} = U_{j \in C'} \sigma(j)$, and $(\{\alpha, \beta\}, -1.5) P_2 (\{\beta\}, 1)$, $(\{\emptyset\}, 1.5) P_3 (\{\gamma\}, 0)$. Thus, z is not *group-envy-free*. ■

In economies with infinitely divisible goods, other concepts of fairness have been proposed. An important one is the Individually Rational solution from equal division, which selects those allocations that all agents prefer to equal division. Moulin (1990) proposes the *Equal Split Guarantee* solution as an extension of this solution for economies with indivisible goods.

For a formal definition, we need some more notation.

Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, for each $i \in Q$, let $e^i = (Q, \Omega, M; R_Q^i) \in \mathcal{E}$ be such that $R_j^i = R_i$ for all $j \in Q$.

The economy e^i is obtained from the economy e by imagining that all agents have the same preferences as agent i . For such an economy we define;

$$E(e^i) = \{z^i \in P(e^i) / \forall j, k \in Q, z_{j1}^i z_k^i\}$$

The set $E(e^i)$ is the set of *Pareto efficient* allocations at which each agent is indifferent between what he receives and what the others receive. Since $R_j^i = R_i$ it is easy to check that $E(e^i)$ is non-empty and for all $z^i, \bar{z}^i \in E(e^i)$, z^i, \bar{z}^i are Pareto indifferent.

The Equal Split Guarantee solution, E: given $e = (Q, \Omega, M; R_Q)$

$$E(e) = \{z \in Z(e) / \forall i \in Q, \forall z^i \in E(e^i), z_i R_i z^i\}$$

In the quasi-linear economies, $z = (\sigma, m) \in E(e)$ if and only if.

$$v_{1\sigma(1)} + m_{\sigma(1)} \geq \frac{\sum_{j \in Q} v_{1\tau(j)} + M}{q} \quad \text{for all } i \in Q, \tau \in \Sigma(e^1)$$

This solution can be empty as the following example shows.

Example 1. Let $e = (Q, \Omega, M; R_Q)$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$, $v_{1\alpha} = 4$, $v_{1\beta} = 4$, $v_{1(\alpha, \beta)} = 10$, and $v_{2\alpha} = 10$, $v_{2\beta} = 10$, $v_{2(\alpha, \beta)} = 14$. It is easy to check that for this economy there is no feasible allocation which satisfies the above inequality.

The next proposition gives a necessary and sufficient condition for the non-emptiness of the solution.

Proposition 2.5. Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}$, $E(e) \neq \emptyset$ if and only if there is a partition $\mathcal{P} \in \mathcal{P}(\Omega, q)$ and an assignment $\sigma: Q \rightarrow \mathcal{P}$ such that

$$\sum_{i \in Q} v_{i\sigma(i)} \geq \frac{\sum_{i \in Q} (\sum_{j \in Q} v_{i\tau^1(j)})}{q} \quad \text{where } \tau^1 \in \Sigma(e^1)$$

Proof. The straightforward proof is omitted.

The above condition is always satisfied if all agents have **separable** preferences, that is, each agent values a set of objects as the sum of the values of each of the objects in that set.

In traditional economies, the *No-Envy* solution and the *Equal Split Guarantee* solution are related by inclusion. Furthermore, both solutions

coincide in economies with two agents. In general, these results do not hold as is showed in Propositions 2.6 and 2.7. However, it is easy to verify that those relations are maintained if preferences are *separable*.

Proposition 2.6. The *No-Envy* solution and the *Equal Split Guarantee* solution are not related by inclusion. The *Equal Split Guarantee* solution and the *Group-No-Envy* solution are not related either.

Proof. Let $e = (Q, \Omega, M; R_Q)$ be such that $Q = \{1, 2, 3\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$,
 $v_{1\alpha} = 8$, $v_{1\beta} = 2$, $v_{1(\alpha, \beta)} = 12$; $v_{2\alpha} = 2$, $v_{2\beta} = 4$, $v_{2(\alpha, \beta)} = 8$;
and $v_{3\alpha} = 0$, $v_{3\beta} = 6$, $v_{3(\alpha, \beta)} = 6$.

Let $z = ((\{\alpha\}, 0), (\{\emptyset\}, 2), (\{\beta\}, -2))$. We have that $z \in N(e)$ and $z \in G(e)$, but $z \notin E(e)$. Let $z' = ((\{\alpha\}, -4), (\{\emptyset\}, 6), (\{\beta\}, -2))$. We have that $z' \in E(e)$, but $z' \notin N(e)$ and $z' \notin G(e)$. ■

However, the *Equal Split Guarantee* solution is a subsolution of the *No-Envy* solution for two agent economics.

Proposition 2.7. For any economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ with $|Q| = 2$, $E(e) \subseteq N(e)$. However, there exists $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ with $|Q| = 2$ such that $E(e) \neq N(e)$.

Proof. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, and $z = (\sigma, m) \in E(e)$ be given. Suppose that $z \notin N(e)$. Without loss of generality, suppose that agent 1 envies agent 2.

Thus,

$$v_{1\sigma(2)} + m_{\sigma(2)} > v_{1\sigma(1)} + m_{\sigma(1)}$$

Since $z \in E(e)$, for $\tau \in \Sigma(e^1)$,

$$v_{1\sigma(1)} + m_{\sigma(1)} \geq \frac{v_{1\tau(1)} + v_{1\tau(2)} + M}{2}$$

Since $\tau \in \Sigma(e^1)$,

$$v_{1\tau(1)} + v_{1\tau(2)} \geq v_{1\sigma(1)} + v_{1\sigma(2)}.$$

Therefore,

$$v_{1\sigma(2)} + m_{\sigma(2)} > v_{1\sigma(1)} + m_{\sigma(1)} \geq \frac{v_{1\sigma(1)} + v_{1\sigma(2)} + M}{2}$$

Thus,

$$m_{\sigma(1)} \geq \frac{-v_{1\sigma(1)} + v_{1\sigma(2)} + M}{2}; \quad m_{\sigma(2)} > \frac{v_{1\sigma(1)} - v_{1\sigma(2)} + M}{2}$$

Then, $m_{\sigma(1)} + m_{\sigma(2)} > M$, in contradiction with the feasibility of z .

This concludes the first part of the proposition. In order to prove the second part let $e = (Q, \Omega, M; R_Q)$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$, $v_{1\alpha} = 8$, $v_{1\beta} = 2$, $v_{1(\alpha, \beta)} = 12$; and $v_{2\alpha} = 2$, $v_{2\beta} = 4$, $v_{2(\alpha, \beta)} = 6$. For this economy, $z = ((\alpha, -3), (\beta, 3)) \in N(e)$, but $z \notin E(e)$. ■

It should be remarked that Propositions 2.6 and 2.7 also hold if we consider the intersection of the solutions involved with the *Pareto* solution.

Another solution concept that is related with the *No-Envy* and the *Equal Split Guarantee* solution is the *Equal Income Walrasian* solution. For a formal definition, we introduce some more notation.

Let $p = (p_\alpha, p_\beta, \dots) \in \mathbb{R}^{|\Omega|}$ denote prices for the indivisible goods. Given $\mathcal{A} = \{\alpha_1, \dots, \alpha_k\} \in \mathcal{P}(\Omega)$, let $p_{\mathcal{A}} = p_{\alpha_1} + \dots + p_{\alpha_k}$.

The Equal Income Walrasian solution, WI: given $e \in \mathcal{E}$, $z = (\sigma, m) \in \text{WI}(e)$, if $z \in Z(e)$ and a price vector $p \in \mathbb{R}^{|\Omega|}$ exists such that,

- (i) if $(\mathcal{A}, m_{\mathcal{A}}) P_i(\sigma(i), m_{\sigma(i)})$, then $p_{\mathcal{A}} + m_{\mathcal{A}} > p_{\sigma(i)} + m_{\sigma(i)}$
- (ii) $p_{\sigma(i)} + m_{\sigma(i)} = p_{\sigma(j)} + m_{\sigma(j)}$ for all $i, j \in Q$.

Proposition 2.8. For all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, $\text{WI}(e) \subseteq \text{EP}(e) \cap \text{NP}(e)$.

Proof. Clearly, the *Equal Income Walrasian* solution selects allocations that are *Pareto efficient* and *envy-free*. It remains to be proven that it is a subsolution of the *Equal Split Guarantee* solution. Let $e \in \mathcal{E}$, and $z = (\sigma, m) \in \text{WI}(e)$ be given. Suppose that $z \notin \text{EP}(e)$. Thus, an agent $i \in Q$ exists such that $z_i^1 P_i z_i^1$ for all $z_i^1 \in E(e^1)$. Suppose that $z_i^1 = (\tau, \bar{m})$. Since $z_i^1 I_i z_j^1$ for all $j \in Q$, $(\tau(j), \bar{m}_{\tau(j)}) P_i(\sigma(i), m_{\sigma(i)})$ for all $j \in Q$. Since $z \in \text{WI}(e)$, $p_{\tau(j)} + \bar{m}_{\tau(j)} > p_{\sigma(i)} + m_{\sigma(i)}$ for all $j \in Q$. Since $p_{\sigma(i)} + m_{\sigma(i)} = p_{\sigma(j)} + m_{\sigma(j)}$, then $\sum_{j \in Q} p_{\tau(j)} + \bar{m}_{\tau(j)} > \sum_{j \in Q} p_{\sigma(j)} + m_{\sigma(j)}$ which is a contradiction because $\sum_{j \in Q} \bar{m}_{\tau(j)} = \sum_{j \in Q} m_{\sigma(j)} = M$, and $\sum_{j \in Q} p_{\tau(j)} = \sum_{j \in Q} p_{\sigma(j)}$ because $\bigcup_{j \in Q} \tau(j) = \Omega = \bigcup_{j \in Q} \sigma(j)$. ■

As we have mentioned in the introduction (Proposition 1.2) in traditional economies the *Equal Income Walrasian* solution is equivalent to the *No-Envy* solution. By Propositions 2.6 and 2.8 we can conclude that this equivalence does not hold in our model.

We close this section by studying the relationship between the *Equal Income Walrasian* solution and the *Group-No-Envy* solution.

Proposition 2.9. For all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, $WI(e) \subseteq G(e)$.

Proof. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ and $z = (\sigma, m) \in WI(e)$ be given. Suppose that $z \notin G(e)$. Then, coalitions $C, C' \subseteq Q$ with $|C| = |C'|$, and an allocation $z' = (\tau, m') \in Z(e)$ exist such that, $U_{i \in C} \tau(i) = U_{j \in C'} \sigma(j)$, $\sum_{i \in C} m'_{\tau(i)} = \sum_{j \in C'} m_{\sigma(j)}$, and $z'_i R_i z_i$ for all $i \in C$ with at least a strict preference relation for some $i \in C$. Because $z \in WI(e)$, $p_{\tau(i)} + m'_{\tau(i)} \geq p_{\sigma(i)} + m_{\sigma(i)}$ for all $i \in C$ with at least a strict inequality. Since $p_{\sigma(i)} + m_{\sigma(i)} = p_{\sigma(j)} + m_{\sigma(j)}$ for all $i, j \in Q$, $\sum_{i \in C} p_{\tau(i)} + m'_{\tau(i)} > \sum_{j \in C'} p_{\sigma(j)} + m_{\sigma(j)}$. This is a contradiction because $\sum_{i \in C} m'_{\tau(i)} = \sum_{j \in C'} m_{\sigma(j)}$, and $U_{i \in C} \tau(i) = U_{j \in C'} \sigma(j)$, therefore $\sum_{i \in C} p_{\tau(i)} = \sum_{j \in C'} p_{\sigma(j)}$. Thus, for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, $WI(e) \subseteq G(e)$. Moreover, by Propositions 2.6 and 2.8 we can conclude that the *Equal Income Walrasian* solution is a proper subsolution of the *Group-No-Envy* solution. ■

To summarize, we present in Figure 1 the relations among all the solutions discussed in this section in the traditional model. Figure 2 presents the relation among such solutions in the general model.

traditional models

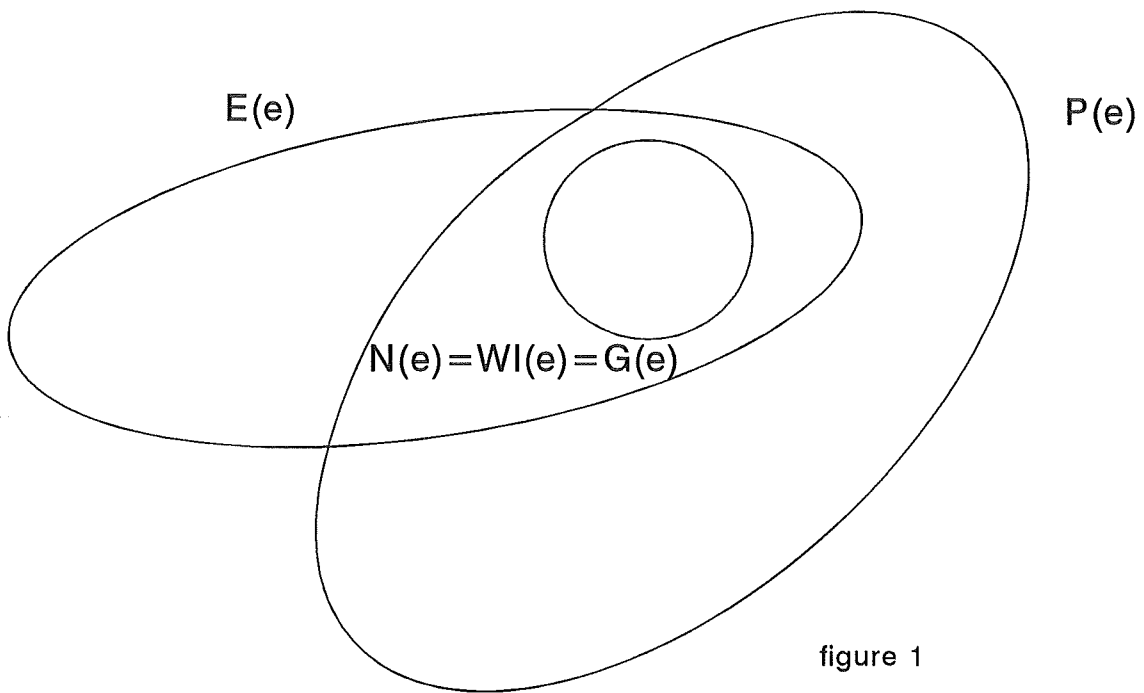


figure 1

general models

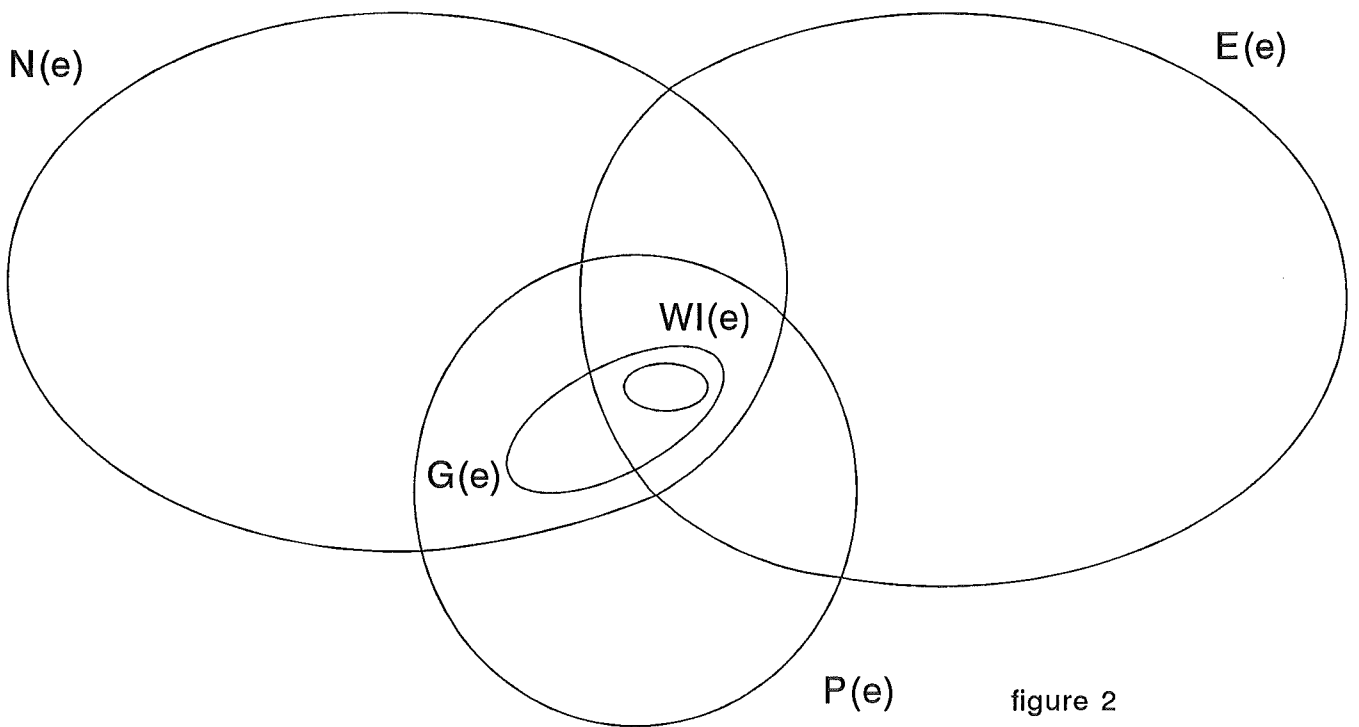


figure 2

3. CONSISTENCY.

The purpose of this section is to analyze the *No-Envy* solution in the light of *consistency* properties. We show that most of the results obtained with traditional economies do not hold in our models, but some of them can be generalized.

A solution φ is *consistent* if whenever an allocation is a good recommendation, according to φ , for an economy, the restriction of this allocation to any subgroup of agents is also a good recommendation, according to φ , for the problem of allocating the resources receive by this subgroup.

Consistency. The solution φ is **consistent** if for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, for all $Q' \subset Q$, for all $z = (\sigma, m) \in \varphi(e)$, $z_{Q'} = (\sigma(i), m_{\sigma(i)})_{i \in Q'} \in \varphi(r_Q^z(e))$, where

$$r_Q^z(e) = (Q', \bigcup_{i \in Q'} \sigma(i), \sum_{i \in Q'} m_{\sigma(i)}; R_{Q'}).$$

We call $r_Q^z(e)$ the **reduced economy of e with respect to z and Q'** .

Neutrality. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ be given, an allocation $z' \in Z(e)$ is obtained by an indifference permutation from $z \in Z(e)$ if there is a permutation $\pi: Q \rightarrow Q$ such that for all $i \in Q$, $z'_i = z_{\pi(i)}$, and $z'_i I_i z_i$. This equivalence relation is denoted by \approx . A solution φ satisfies **neutrality** if for all $e = (Q, A, M; R_Q) \in \mathcal{E}$, for all $z \in \varphi(e)$ and for all $z' \approx z$, $z' \in \varphi(e)$.

In our model, the difficulty raised in Proposition 1.4. does not hold.

Proposition 3.1. There are proper subsolutions of the *No-Envy* and *Pareto* solution which satisfy *consistency* and *neutrality*.

Proof. It is easy to check that the *Group-No-Envy* solution and the *Equal Income Walrasian* solution satisfy *consistency* and *neutrality*. Furthermore, as we have shown in the previous section, both solutions are proper subsolutions of the *No-Envy* and *Pareto* solution. ■

The above mentioned solutions are not the only ones satisfying *consistency* and *neutrality*. There is a class of solutions that are based on lexicographic⁽¹⁾ orders on $\mathbb{P}(\Omega, q)$ and are *consistent* subsolutions of the *No-Envy* and *Pareto* solution. The next paragraph describes a subsolution in this class.

The solution ψ ; given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q_1}$, let $q = |Q|$, $w = |\Omega|$, and $0 \leq k \leq w$. For any $\sigma \in \Sigma(e)$, let n_σ^k be the number of agents who receive k objects in the assignment σ . An allocation $z = (\sigma, m) \in \psi(e)$ if $z \in NP(e)$ and for any other $\tau \in \Sigma(e)$, $(n_\sigma^0, n_\sigma^{w-1}, \dots, n_\sigma^1) \leq_L (n_\tau^0, n_\tau^{w-1}, \dots, n_\tau^1)$, where \leq_L is the lexicographic order.

(1) Given two vectors $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $(x_1, x_2, \dots, x_n) \leq_L (y_1, y_2, \dots, y_n)$ if either $x_1 < y_1$ or $x_1 = y_1$ and $x_2 < y_2$ or $x_2 = y_2$ and $x_3 < y_3$, and so on.

The solution ψ chooses those *envy-free* and *Pareto efficient* allocations in which the objects are more evenly distributed among the agents. In other words, between two *envy-free* and *Pareto efficient* allocations, ψ chooses the one at which few agents receive no objects. If, in both allocations, the same number of agents receive no objects, then ψ chooses the one which few agents receive all the objects minus one. And so on.

Notice that, in traditional economies, the solution ψ is the *No-Envy* solution, however, in our context this solution is more restricted.

Proposition 3.2. The solution ψ satisfies *consistency* and *neutrality* in \mathcal{E}_{q1} .

Proof. It is easy to see that the solution ψ satisfies *neutrality*. We have to prove that it is *consistent*. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, $z = (\sigma, m) \in \psi(e)$ be given. Let $Q' \subseteq Q$. Since NP is *consistent* $z_{Q'} \in NP(r_{Q'}^z(e))$. Let $\Omega' = \bigcup_{i \in Q'} \sigma(i)$ and let $w' = |\Omega'|$. Suppose that an assignment $\tau' \in \Sigma(r_{Q'}^z(e))$ exists such that $(n_{\sigma_{Q'}}^0, n_{\sigma_{Q'}}^w, \dots, n_{\sigma_{Q'}}^1) >_L (n_{\tau'}^0, n_{\tau'}^w, \dots, n_{\tau'}^1)$. Let τ be an assignment in the economy e such that $\tau_{Q'} = \tau'$ and $\tau_{Q \setminus Q'} = \sigma_{Q \setminus Q'}$. Since $\tau', \sigma_{Q'} \in \Sigma(r_{Q'}^z(e))$, $\sum_{i \in Q'} v_i \sigma(i) = \sum_{i \in Q'} v_i \tau'(i)$. Since $\tau_{Q \setminus Q'} = \sigma_{Q \setminus Q'}$, $\sum_{i \in Q} v_i \sigma(i) = \sum_{i \in Q} v_i \tau(i)$. Therefore, $\tau \in \Sigma(e)$, and $(n_{\sigma}^0, n_{\sigma}^w, \dots, n_{\sigma}^1) >_L (n_{\tau}^0, n_{\tau}^w, \dots, n_{\tau}^1)$, which is a contradiction because $z = (\sigma, m) \in \psi(e)$. Thus, the solution ψ satisfies *consistency*. ■

Other properties related with *consistency* have been studied in traditional economies in relation to the *No-Envy* solution.

The following definition is a weakening of *consistency* obtained by applying it only to subgroups of cardinality two. Formally,

Bilateral consistency. The solution φ is **bilaterally consistent** if, for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, for all $Q' \subset Q$ with $|Q'| = 2$, for all $z \in \varphi(e)$, $z_{Q'} \in \varphi(r_{Q'}^z(e))$.

Since *bilateral consistency* is weaker than *consistency*, proper subsolutions of the *No-Envy* solution exist which satisfy the property (the *Group-No-Envy* solution, the *Equal Income Walrasian* solution, the solution ψ, \dots). In traditional economies all such subsolutions coincide with the *No-Envy* solution for economies with two agents. This result does not hold in our model.

Proposition 3.3. There exists $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ with $|Q| = 2$ such that $WI(e) \neq NP(e)$ and $\psi(e) \neq NP(e)$.

Proof. By Proposition 2.8, for all $e \in \mathcal{E}$, $WI(e) \subseteq NP(e) \cap EP(e)$. By Proposition 2.7 there exists $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ with $|Q| = 2$ such that $EP(e) \neq NP(e)$. Therefore, for such an economy, $WI(e) \neq NP(e)$. In order to prove the second inequality, let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$, $v_{1\alpha} = 3$, $v_{1\beta} = 1$, $v_{1(\alpha, \beta)} = 5$; $v_{2\alpha} = 1$, $v_{2\beta} = 2$, $v_{2(\alpha, \beta)} = 4$. The set of efficient assignments for the economy e is,

$$\Sigma(e) = \{ \sigma, \tau \ / \ \sigma(1) = \{\alpha, \beta\}, \sigma(2) = \{\emptyset\}, \tau(1) = \{\alpha\}, \tau(2) = \{\beta\} \}$$

Let $z = ((\alpha, \beta), -2), (\emptyset, 2) \in Z(e)$. Clearly $z \in NP(e)$, but $z \notin \psi(e)$ because $(n_\sigma^0, n_\sigma^2, n_\sigma^1) >_L (n_\tau^0, n_\tau^2, n_\tau^1)$. Thus, $\psi(e) \subset NP(e)$. ■

Although Proposition 1.5 does not hold in our model, we can obtain a generalization of this result by adding a weak property called *Property B*. In order to introduce this result we need some more notation.

Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, let $D(e)$ be the set of efficient partitions, that is

$$D(e) = \{ \mathcal{P} \in \mathcal{P}(\Omega, q) / \exists \sigma: Q \rightarrow \mathcal{P} \text{ s.t. } \sigma \in \Sigma(e) \}$$

Given a solution φ , let $D_\varphi(e)$ be the set of φ -efficient partitions, that is,

$$D_\varphi(e) = \{ \mathcal{P} \in D(e) / \exists \sigma: Q \rightarrow \mathcal{P} \text{ s.t. } (\sigma, m) \in \varphi(e) \text{ for some } m \in \mathbb{R}^q \}$$

Notice that $D_{NP}(e) = D(e)$.

For each $\mathcal{P} \in D_\varphi(e)$, let $\varphi_{\mathcal{P}}(e) = \{ z = (\sigma, m) \in \varphi(e) / \sigma(Q) = \mathcal{P} \}$.

Property B states that if an economy is extended in such a way that a new agent and a new object is added, and such that any new efficient partition is the union of the new object with an initial efficient partition, then, if the new efficient partition is the union of the new object with an initial φ -efficient partition, the partition itself is a φ -efficient partition for the new economy.

Property B. A *Pareto efficient* solution satisfies *Property B* if for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$, and for all $e' = (Q', \Omega', M'; R'_Q) \in \mathcal{E}$ such that $Q' = Q \cup \{i_0\}$, $\Omega' = \Omega \cup \{\alpha_0\}$, $R'_Q = R_Q$, all $\mathcal{P}' \in D(e')$ is such that $\mathcal{P}' = \mathcal{P} \cup \{\alpha_0\}$ with $\mathcal{P} \in D(e)$. Then if $\mathcal{P}' = \mathcal{P} \cup \{\alpha_0\} \in D(e')$ with $\mathcal{P} \in D_\varphi(e)$, $\mathcal{P}' \in D_\varphi(e')$.

The *No-Envy* and *Pareto efficient* solution and the solution ψ are examples of solutions that satisfy *Property B*. The *Equal Income Walrasian* solution does not satisfy the property. We prove it in an indirect way using Proposition 3.4, so we give the proof of this result in Proposition 3.5.

Proposition 3.4. Let φ be a subsolution of the *No-Envy* and *Pareto efficient* solution. If φ satisfies *bilateral consistency*, *neutrality*, and *Property B*, then, for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ such that $|Q| = 2$, $\varphi(e) = \bigcup_{\mathcal{P} \in D_\varphi(e)} \text{NP}_{\mathcal{P}}(e)$.

Notice that, in traditional economies, all the *Pareto efficient* solutions satisfy *Property B*. Thus, in traditional economies Proposition 3.4 corresponds to Proposition 1.5.

In order to prove the proposition we need the following lemma.

Lemma 3.1. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, let $|Q| = 2$, let $\mathcal{P} \in D(e)$, and let $z = (\sigma, m) \in \text{NP}_{\mathcal{P}}(e)$. Then, an economy $e' = (Q', \Omega', M'; R'_Q)$ and an allocation $z' \in \text{NP}_{\mathcal{P}}(e')$ exist such that $e = r_Q^{z'}(e')$ and $z'_Q = z$. Moreover, for all $z'' \in \text{NP}_{\mathcal{P}}(e')$, $z'' \approx z'$.

In order to simplify the exposition, we give the proof of the Lemma in the Appendix.

Proof of Proposition 3.4. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ be such that $|Q| = 2$, let $\mathcal{P} \in D_\varphi(e)$, and let $z \in NP_{\mathcal{P}}(e)$ be given. Let $e' = (Q', \Omega', M'; R_{Q'}) \in \mathcal{E}_{q1}$ and $z' \in NP_{\mathcal{P}}(e')$ be as described in Lemma 3.1. Since φ satisfies *property B*, $\mathcal{P}' \in D_\varphi(e')$. Let $z'' \in \varphi_{\mathcal{P}'}(e')$. Since $\varphi(e') \subseteq NP(e')$, $z'' \in NP_{\mathcal{P}'}(e')$. By Lemma 3.1, $z'' \simeq z'$, since φ satisfies *neutrality*, $z' \in \varphi_{\mathcal{P}'}(e')$. By *bilateral consistency* $z'_Q = z \in \varphi_{\mathcal{P}}(e)$. Thus, $NP_{\mathcal{P}}(e) \subseteq \varphi_{\mathcal{P}}(e)$ for each $\mathcal{P} \in D_\varphi(e)$, which implies that $\varphi(e) = \bigcup_{\mathcal{P} \in D_\varphi(e)} NP_{\mathcal{P}}(e)$. ■

Proposition 3.5. The *Equal Income Walrasian* solution does not satisfy *Property B*.

Proof. The *Equal Income Walrasian* solution is a subsolution of the *No-envy* and *Pareto efficient* solution which satisfies *consistency* and *neutrality*. If we prove that for economies with two agents it is not true that $WI(e) = \bigcup_{\mathcal{P} \in D_{WI}(e)} NP_{\mathcal{P}}(e)$, then by Proposition 3.4 this solution does not satisfy *Property B*.

Let $e = (Q, \Omega, M; R_Q)$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \beta\}$, $M = 0$,

$$v_{1\alpha} = 4, v_{1\beta} = 2, v_{1(\alpha, \beta)} = 8; v_{2\alpha} = 2, v_{2\beta} = 6, v_{2(\alpha, \beta)} = 8.$$

For this economy,

$$EP(e) = \{((\alpha), m_1), ((\beta), m_2) / 0 \leq m_1 \leq 2\}$$

$$NP(e) = \{((\alpha), m_1), ((\beta), m_2) / -1 \leq m_1 \leq 2\}$$

Since $WI(e) \subseteq NP(e) \cap EP(e)$, it is not true that $WI(e) = \bigcup_{\mathcal{P} \in D_{WI}(e)} NP_{\mathcal{P}}(e)$. ■

We will now consider a condition that is dual to *consistency*. According to this condition, if for some economy a feasible allocation is such that its restriction to each subgroup of cardinality 2 constitutes a recommendation for the problem of allocating the resources received by this subgroup, then it is in itself a recommendation for the whole economy.

Converse consistency. The solution φ is **conversely consistent** if for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ with $|Q| > 2$, for all $z \in Z(e)$, and, for all $Q' \subset Q$ with $|Q'| = 2$, $z_{Q'} \in \varphi(r_{Q'}^z(e))$, then $z \in \varphi(e)$.

It should be remarked that the *No-Envy* solution satisfies this property. However, the *Pareto efficient* solution is not *conversely consistent* either in traditional economies or in our model. Moreover, in our model none of the intersection of the *No-Envy* solution with the *Pareto* solution, the *Group-No-Envy* solution, and the solution ψ are *conversely consistent*. These negative results are established in the next propositions.

Proposition 3.6 The *Pareto* solution is not *conversely consistent*. the *No-Envy* and *Pareto* solution is not *conversely consistent* either.

Proof. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2, 3\}$, $\Omega = \{\alpha, \beta, \gamma\}$, $M = 0$,
 $v_{1\alpha} = 2$, $v_{1\beta} = 1$, $v_{1\gamma} = 0$, $v_{1(\alpha, \beta)} = 5$, $v_{1(\alpha, \gamma)} = 4$, $v_{1(\beta, \gamma)} = 1$,
 $v_{1\Omega} = 7$; $v_{2\alpha} = 0$, $v_{2\beta} = 3$, $v_{2\gamma} = 3$, $v_{2(\alpha, \beta)} = 3$, $v_{2(\alpha, \gamma)} = 4$, $v_{2(\beta, \gamma)} = 7$,
 $v_{2\Omega} = 10$; $v_{3\alpha} = 1$, $v_{3\beta} = 1$, $v_{3\gamma} = 4$, $v_{3(\alpha, \beta)} = 3$, $v_{3(\alpha, \gamma)} = 6$, $v_{3(\beta, \gamma)} = 6$,
 $v_{3\Omega} = 8$. Let $z = ((\{\alpha\}, 0), (\{\beta\}, 0), (\{\gamma\}, 0)) \in Z(e)$.

Clearly, $((\{\alpha\},0),(\{\beta\},0)) \in P(r_{\{1,2\}}^z(e))$, $((\{\beta\},0),(\{\gamma\},0)) \in P(r_{\{2,3\}}^z(e))$,
 $((\{\alpha\},0),(\{\gamma\},0)) \in P(r_{\{1,3\}}^z(e))$, but $z \notin P(e)$ because in the economy e there
is only one efficient assignment, $\sigma(1) = \{\emptyset\}$, $\sigma(2) = \Omega$, $\sigma(3) = \{\emptyset\}$. Thus
 $z \notin P(e)$. This example can be used to prove that the *No-Envy* and *Pareto*
solution is not *conversely consistent* because

$((\{\alpha\},0),(\{\beta\},0)) \in NP(r_{\{1,2\}}^z(e))$,
 $((\{\beta\},0),(\{\gamma\},0)) \in NP(r_{\{2,3\}}^z(e))$, $((\{\alpha\},0),(\{\gamma\},0)) \in NP(r_{\{1,3\}}^z(e))$,
but $z \notin NP(e)$. ■

Proposition 3.7. The *Group-No-Envy* solution is not *conversely consistent*.

Proof. Let $e \in \mathcal{E}$ be as we have described in Proposition 2.4. Let
 $z = ((\{\alpha\},-1),(\{\beta\},1),(\{\gamma\},1)) \in Z(e)$. For all $Q' \subseteq Q$ with $|Q'| = 2$,
 $z_{Q'} \in G(r_{Q'}^z(e))$ because $z_{Q'}$ is *envy-free* and *Pareto efficient* for e , and for
economies with two agents any *envy-free* and *Pareto efficient* allocation is a
group-envy-free allocation. But, as we have shown in Proposition 2.4,
 $z \notin G(e)$. Therefore, the *Group-No-Envy* solution is not *conversely consistent*. ■

Proposition 1.6 asserts that if a subsolution of the *No-Envy* solution
satisfies *bilateral consistency*, *converse consistency*, and *neutrality*, then it
coincides with the *No-Envy* solution. Adding *Property B*, we obtain a
generalization for our model.

Proposition 3.8. Let φ be a subsolution of the *No-Envy* and *Pareto efficient*
solution. If φ satisfies *bilateral consistency*, *converse consistency*,
neutrality and *Property B*, then for all $e \in \mathcal{E}_{q1}$ $\varphi(e) = \bigcup_{\mathcal{P} \in \mathcal{D}_{\varphi}(e)} NP_{\mathcal{P}}(e)$.

Proof. Since φ satisfies *neutrality*, *Property B* and *bilateral consistency*, by Proposition 3.4, $\varphi(e) = \bigcup_{\mathcal{P} \in D_{\varphi}(e)} NP_{\mathcal{P}}(e)$ for all $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ such that $|Q| = 2$. Suppose, by way of contradiction, that for some economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ such that $|Q| \geq 3$, an allocation $z \in \bigcup_{\mathcal{P} \in D_{\varphi}(e)} NP_{\mathcal{P}}(e) \setminus \varphi(e)$ exists. Since φ satisfies *converse consistency*, there is $Q' \subseteq Q$ with $|Q'| = 2$ such that $z_{Q'} \notin \varphi(r_{Q'}^z(e))$. Since NP satisfies *bilateral consistency* $z_{Q'} \in NP(r_{Q'}^z(e))$. Since $z = (\sigma, m)$ with $\sigma(Q) = \mathcal{P} \in D_{\varphi}(e)$, and φ satisfies *bilateral consistency*, $\sigma(Q') = \mathcal{P}' \in D_{\varphi}(r_{Q'}^z(e))$. Thus, $z_{Q'} \in NP_{\mathcal{P}'}(r_{Q'}^z(e))$. Therefore, $z_{Q'} \in \bigcup_{\mathcal{P}' \in D_{\varphi}(r_{Q'}^z(e))} NP_{\mathcal{P}'}(r_{Q'}^z(e))$. But, by Proposition 3.4, $\bigcup_{\mathcal{P}' \in D_{\varphi}(r_{Q'}^z(e))} NP_{\mathcal{P}'}(r_{Q'}^z(e)) = \varphi(r_{Q'}^z(e))$, which is a contradiction because $z_{Q'} \notin \varphi(r_{Q'}^z(e))$. ■

4. POPULATION MONOTONICITY.

In this Section we study another property involving variations in the number of agents, *population-monotonicity*. This property requires that, whenever new agents appear in the economy to share the same resources, none of the initially present agents benefit. Formally,

Population-Monotonicity. The solution φ satisfies **population-monotonicity** if for all $e = (Q, \Omega, M, R_Q)$ and $e' = (Q', \Omega', M', R_{Q'})$ with $Q \subset Q'$ and $(\Omega, M, R_Q) = (\Omega', M', R_{Q'})$, for all $z \in \varphi(e)$ and $z' \in \varphi(e')$, we have $z_i R_i z'_i$ for all $i \in Q$.

This property was introduced by Thomson (1983) in the context of bargaining (see also Chilchinisky and Thomson (1987), Alkan (1989), Moulin (1990,1992), Tadenuma and Thomson (1993)).

Alkan (1989) shows that *population-monotonicity* is violated by all selections from the *No-Envy* solution in economies with indivisible goods. A simple adaptation of his result shows that the impossibility remains in our context.

If we insist on *population-monotonicity* we should ask whether any selection from the *Pareto* solution satisfying the property exists. In traditional economies Moulin (1992) provides an affirmative answer to this question. He proves that the *Shapley* solution is a Pareto solution that

satisfies *population-monotonicity*. In our model, another impossibility result appears. There is no selection from the *Pareto* solution satisfying *population-monotonicity*. In order to establish this result we need some more notation.

Given an economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}$, for any $S \subseteq Q$, for any $\mathcal{A} \subseteq \Omega$, let $U_S(\mathcal{A}) = \max_{\sigma} \sum_{i \in S} v_{i\sigma(i)}$ where σ is a bijection $\sigma: S \rightarrow \mathcal{P}$, $\mathcal{P} \in \mathcal{P}(\mathcal{A}, |S|)$. We will refer to $U_S(\mathcal{A})$ as the **joint valuation of S on the set \mathcal{A}** .

Given a solution φ and an economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}$, for any $z = (\sigma, m) \in \varphi(e)$ and for any $S \subseteq Q$, let $u_S(z) = \sum_{i \in S} (v_{i\sigma(i)} + m_{\sigma(i)})$.

Notice that if φ is a *Pareto* solution, for any $z \in \varphi(e)$, $u_Q(z) = U_Q(\Omega) + M$.

The following Lemma is needed for the proof of the impossibility result. This Lemma is a direct adaptation for this model of the same result obtained by Moulin (1992) in the context of fair allocation with infinitely divisible goods, monetary transfers and quasi-linear preferences.

Lemma 4.1. Let φ be a *Pareto* solution that satisfies *population-monotonicity*. Then, for any economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}$ such that $Q = \{1, 2, 3, 4\}$, $M = 0$, the following inequality holds⁽²⁾

(2) We consider $M = 0$ for simplicity; the same result can be obtained when M is not zero.

$$U_Q(\Omega) + \sum_{i=1}^4 U_{Q \setminus \{i\}}(\Omega) \leq 2(U_{\{13\}}(\Omega) + U_{\{14\}}(\Omega) + U_{\{23\}}(\Omega) + U_{\{24\}}(\Omega))$$

Proof. Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, for any $S \subseteq Q$ we write $e_S = (S, \Omega, M; R_S)$ and z_S for any allocation in $\varphi(e_S)$.

Since the solution φ satisfies *population-monotonicity*, and it is a *Pareto* solution, for any $z_Q, z_{\{134\}}, z_{\{13\}}, z_{\{14\}}$,

$$u_{\{13\}}(z_{\{134\}}) \leq u_{\{13\}}(z_{\{13\}}) = U_{\{13\}}(\Omega)$$

$$u_{\{14\}}(z_{\{134\}}) \leq u_{\{14\}}(z_{\{14\}}) = U_{\{14\}}(\Omega)$$

Then,

$$u_{\{4\}}(z_{\{134\}}) \geq U_{\{134\}}(\Omega) - U_{\{13\}}(\Omega)$$

$$u_{\{3\}}(z_{\{134\}}) \geq U_{\{134\}}(\Omega) - U_{\{14\}}(\Omega)$$

Thus,

$$u_{\{1\}}(z_Q) \leq u_{\{1\}}(z_{\{134\}}) \leq U_{\{13\}}(\Omega) + U_{\{14\}}(\Omega) - U_{\{134\}}(\Omega)$$

In the same way, we obtain

$$u_{\{2\}}(z_Q) \leq u_{\{2\}}(z_{\{234\}}) \leq U_{\{23\}}(\Omega) + U_{\{24\}}(\Omega) - U_{\{234\}}(\Omega)$$

$$u_{\{3\}}(z_Q) \leq u_{\{3\}}(z_{\{123\}}) \leq U_{\{13\}}(\Omega) + U_{\{23\}}(\Omega) - U_{\{123\}}(\Omega)$$

$$u_{\{4\}}(z_Q) \leq u_{\{4\}}(z_{\{124\}}) \leq U_{\{14\}}(\Omega) + U_{\{24\}}(\Omega) - U_{\{124\}}(\Omega)$$

Since the solution φ is a *Pareto* solution, $u_Q(z_Q) = U_Q(\Omega)$. Summing up the above inequalities we obtain that,

$$U_Q(\Omega) + \sum_{i=1}^4 U_{Q \setminus \{i\}}(\Omega) \leq 2(U_{\{13\}}(\Omega) + U_{\{14\}}(\Omega) + U_{\{23\}}(\Omega) + U_{\{24\}}(\Omega))$$

which completes the proof. ■

Proposition 4.1. There is no subsolution of the Pareto solution satisfying *population-monotonicity*.

Proof. Let $e = (Q, \Omega, M; R_Q)$ be such that $Q = \{1, 2, 3, 4\}$, $\Omega = \{\alpha, \beta, \gamma\}$, $M = 0$, and

$$v_{1\alpha} = 20, v_{1\beta} = 5, v_{1\gamma} = 80, v_{1(\alpha, \beta)} = 30, v_{1(\alpha, \gamma)} = 60, v_{1(\beta, \gamma)} = 75, v_{1(\alpha, \beta, \gamma)} = 65;$$

$$v_{2\alpha} = 5, v_{2\beta} = 20, v_{2\gamma} = 20, v_{2(\alpha, \beta)} = 25, v_{2(\alpha, \gamma)} = 15, v_{2(\beta, \gamma)} = 15, v_{2(\alpha, \beta, \gamma)} = 35;$$

$$v_{3\alpha} = 20, v_{3\beta} = 5, v_{3\gamma} = 5, v_{3(\alpha, \beta)} = 15, v_{3(\alpha, \gamma)} = 15, v_{3(\beta, \gamma)} = 10, v_{3(\alpha, \beta, \gamma)} = 35;$$

$$v_{4\alpha} = 10, v_{4\beta} = 80, v_{4\gamma} = 30, v_{4(\alpha, \beta)} = 70, v_{4(\alpha, \gamma)} = 40, v_{4(\beta, \gamma)} = 50, v_{4(\alpha, \beta, \gamma)} = 70.$$

If a *Pareto* solution satisfies *population-monotonicity*, Lemma 4.1 holds.

But, in this example,

$$U_Q(\Omega) = 180, U_{\langle 234 \rangle}(\Omega) = 120, U_{\langle 134 \rangle}(\Omega) = 180, U_{\langle 124 \rangle}(\Omega) = 165,$$

$$U_{\langle 123 \rangle}(\Omega) = 120, U_{\langle 13 \rangle}(\Omega) = 95, U_{\langle 14 \rangle}(\Omega) = 150, U_{\langle 23 \rangle}(\Omega) = 35, U_{\langle 24 \rangle}(\Omega) = 95.$$

Thus,

$$U_Q(\Omega) + \sum_{i=1}^4 U_{Q \setminus \{i\}}(\Omega) = 765 > 750 = 2(U_{\langle 13 \rangle}(\Omega) + U_{\langle 14 \rangle}(\Omega) + U_{\langle 23 \rangle}(\Omega) + U_{\langle 24 \rangle}(\Omega))$$

Therefore, such a subsolution does not exist. ■

In the context of fair division with money, a possibility result is obtained if goods are *substitutes* (Moulin (1992)). In traditional economies, *substitutability* is a consequence of the fact that one agent can not receive more than one indivisible good (see Shapley (1962)). In our context this condition does not hold in general, and this is what is behind Proposition 4.1. In what follows, we show that if *substitutability* is imposed, positive results are possible.

Given an economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ and given $S \subseteq Q$, the joint valuation U_S satisfies *substitutability* if for all $A \subseteq \Omega$, for all $\alpha, \beta \in \Omega$,

$$U_S(\mathcal{A} \cup \{\alpha, \beta\}) - U_S(\mathcal{A} \cup \{\alpha\}) \leq U_S(\mathcal{A} \cup \{\beta\}) - U_S(\mathcal{A})$$

Let \mathcal{E}_{qls}^+ be the class of quasi-linear economies such that all the joint valuations satisfy *substitutability*, and $M \geq 0$.

Notice that the economy described in Proposition 4.1 is not in this class because $U_3(\{\alpha, \beta, \gamma\}) - U_3(\{\beta, \gamma\}) = 35 - 10 > 15 - 5 = U_3(\{\alpha, \beta\}) - U_3(\{\beta\})$.

Given an economy $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{ql}^+$, let (Q, v) be the **cooperative game** defined by $v(S) = U_S(\Omega) + M$ for all $S \subseteq Q$.

The Shapley solution, Sh: given $e = (Q, \Omega, M; R_Q)$,

$$\text{Sh}(e) = \{z = (\sigma, m) \in Z(e) / \sigma \in \Sigma(e), \text{ and } m_i = \text{sh}_i(Q, v) - v_{i\sigma(i)} \text{ for all } i \in Q\}$$

where $\text{sh}_i(Q, v)$ is the **Shapley value of agent i in the game (Q, v)** .

This solution is essentially single-valued, that is, for any $z, z' \in \text{Sh}(e)$, $z_i = z'_i$ for all $i \in Q$.

Proposition 4.2. The Shapley solution defined on \mathcal{E}_{qls}^+ is a *Pareto solution* which satisfies *population-monotonicity*.

Proof. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{qls}^+$ be given. We divide the proof into steps.

Step 1. For all $S \subseteq Q$, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \Omega$ such that $\mathcal{A} \subseteq \mathcal{B}$

$$U_S(\mathcal{B} \cup \mathcal{C}) - U_S(\mathcal{B}) \leq U_S(\mathcal{A} \cup \mathcal{C}) - U_S(\mathcal{A}) \quad [1]$$

This inequality is a consequence of *substitutability*. For completeness we give the proof in the appendix.

Step 2. For all $S \subseteq Q$, for all $i, j \in Q$,

$$U_{SU(i,j)}(\Omega) - U_{SU(i)}(\Omega) \leq U_{SU(j)}(\Omega) - U_S(\Omega) \quad [2]$$

Let σ be an efficient assignment of Ω among $SU(i, j)$.

Thus,

$$U_{SU(i,j)}(\Omega) = \sum_{k \in SU(i,j)} v_{k\sigma(k)} = U_S(\sigma(S)) + U_i(\sigma(i)) + U_j(\sigma(j))$$

Let $\mathcal{A} = \sigma(S)$, $\mathcal{B} = \sigma(S) \cup \sigma(i)$, $\mathcal{C} = \sigma(j)$. By step 1,

$$U_S(\Omega) - U_S(\sigma(S) \cup \sigma(i)) \leq U_S(\sigma(S) \cup \sigma(j)) - U_S(\sigma(S))$$

Thus,

$$U_{SU(i,j)}(\Omega) + U_S(\Omega) \leq U_S(\sigma(S) \cup \sigma(i)) + U_S(\sigma(S) \cup \sigma(j)) + U_i(\sigma(i)) + U_j(\sigma(j))$$

Since,

$$U_S(\sigma(S) \cup \sigma(j)) + U_i(\sigma(i)) \leq U_{SU(i)}(\Omega)$$

and,

$$U_S(\sigma(S) \cup \sigma(i)) + U_j(\sigma(j)) \leq U_{SU(j)}(\Omega)$$

we obtain,

$$U_{SU(i,j)}(\Omega) + U_S(\Omega) \leq U_{SU(i)}(\Omega) + U_{SU(j)}(\Omega)$$

Step 3. The game (Q, v) is concave, that is, for all $S \subseteq S'$, all $i \notin S'$

$$v(S' \cup \{i\}) - v(S') \leq v(SU(i)) - v(S) \quad [3]$$

Since $v(S) = U_S(\Omega) + M$, to prove [3] is equivalent to proving the following,

$$U_{S' \cup \{i\}}(\Omega) - U_{S'}(\Omega) \leq U_{S \cup \{i\}}(\Omega) - U_S(\Omega)$$

and this inequality is an immediate consequence of [2].

Step 4. For all $e = (Q, \Omega, M; R_Q)$, for all $e' = (Q', \Omega, M; R_{Q'})$ such that $Q \subseteq Q'$,

$$sh_i(Q', v) \leq sh_i(Q, v) \text{ for all } i \in Q. \quad [4]$$

Since the game (Q', v) is concave (Step 3), by a result of Sprumont (1990) the *Shapley value* satisfies [4] (see also Moulin (1992) for an intuition of this result).

By [4], the *Shapley solution* satisfies *population-monotonicity*. ■

FINAL REMARKS.

In this paper we have studied the problem of fair allocation in economies with indivisible goods without imposing the restriction that one agent receives at most one indivisible good. As we have seen, this restriction imposes a great deal of structure on the economy, in such a way, that when dropped, most results obtained in traditional models do not hold. This suggests that the reason that those results hold in traditional models, is not the presence of indivisibilities, but the aforementioned restriction. When the traditional model is generalized to allow for the consumption of several goods by one agent, results became much more like those in models with divisible goods. (Think, for instance, of the results obtained with respect to population monotonicity).

APPENDIX

Proof of Lemma 2.1. Firstly we prove that the allocation $(\sigma, m) \in Z(e)$ such that $\sum_{i \in Q} v_{i\sigma(i)} \geq \sum_{i \in Q} v_{i\tau(i)}$ for any other assignment τ is *Pareto efficient* for e . Suppose that (σ, m) is not *Pareto efficient* for e . Thus, an allocation $(\tau, m') \in Z(e)$ exists such that $(\tau(i), m'_i) R_i(\sigma(i), m_i)$ for all $i \in Q$ with at least a strict preference relation. Since $e \in \mathcal{E}_{q1}$, $(\sigma(i), m_i) I_i(\emptyset, v_{i\sigma(i)} + m_i)$ for all $i \in Q$, and $(\tau(i), m'_i) I_i(\emptyset, v_{i\tau(i)} + m'_i)$ for all $i \in Q$. So, $(\emptyset, v_{i\tau(i)} + m'_i) R_i(\emptyset, v_{i\sigma(i)} + m_i)$ for all $i \in Q$ with at least a strict preference relation. Since preferences are strictly increasing in money, $v_{i\tau(i)} + m'_i \geq v_{i\sigma(i)} + m_i$ for all $i \in Q$ with at least a strict inequality. Thus,

$$\sum_{i \in Q} v_{i\tau(i)} + \sum_{i \in Q} m'_i > \sum_{i \in Q} v_{i\sigma(i)} + \sum_{i \in Q} m_i$$

By feasibility, $\sum_{i \in Q} m'_i = \sum_{i \in Q} m_i$, then, $\sum_{i \in Q} v_{i\tau(i)} > \sum_{i \in Q} v_{i\sigma(i)}$ which is a contradiction.

Secondly we prove that if $(\sigma, m) \in Z(e)$ is a *Pareto efficient* allocation for e , $\sum_{i \in Q} v_{i\sigma(i)} \geq \sum_{i \in Q} v_{i\tau(i)}$ for any other assignment τ . Suppose that the above inequality does not hold. Then, an assignment τ exists such that $\sum_{i \in Q} v_{i\tau(i)} > \sum_{i \in Q} v_{i\sigma(i)}$. Let $Q_1 \subseteq Q$ be such that $v_{i\tau(i)} \geq v_{i\sigma(i)}$ for all $i \in Q_1$, and let $Q_2 \subseteq Q$ be such that $v_{i\tau(i)} \leq v_{i\sigma(i)}$. Clearly, Q_1 and Q_2 are not empty. Let (τ, m') be such that, for all $i \in Q_2$, $m'_i = v_{i\sigma(i)} - v_{i\tau(i)} + m_i$, and for all $i \in Q_1$, $m'_i = d_i + m_i$ where d_i is such that $v_{i\tau(i)} + d_i \geq v_{i\sigma(i)}$ for all $i \in Q_1$ with at least a strict inequality, and

$\sum_{i \in Q_1} d_i + \sum_{i \in Q_2} (v_{i\sigma(i)} - v_{i\tau(i)}) = 0$. These d_i exist because

$$\sum_{i \in Q_2} (v_{i\tau(i)} - v_{i\sigma(i)}) > \sum_{i \in Q_1} (v_{i\sigma(i)} - v_{i\tau(i)}).$$

It is easy to check that the allocation (τ, m') is feasible and $(\tau(i), m'_i) R_i(\sigma(i), m_i)$ for all $i \in Q$ with at least a strict inequality in contradiction with the *Pareto efficiency* of (σ, m) . ■

Proof of Proposition 2.2. Given $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$, let $q = |Q|$, let $\mathcal{P} \in \mathcal{P}(\Omega, q)$, and let σ be such that $\sum_{i \in Q} v_{i\sigma(i)} \geq \sum_{i \in Q} v_{i\tau(i)}$ for any other assignment $\tau: Q \rightarrow \mathcal{P}$.

By the Duality Theorem (Gale, 1960), a q -vector $(x_A)_{A \in \mathcal{P}}$ exists such that

$$v_{i\sigma(i)} + x_{\sigma(i)} \geq v_{iA} + x_A \text{ for all } A \in \mathcal{P}, \text{ for all } i \in Q$$

Let $d = \frac{M - \sum_{i \in Q} x_i}{q}$, and let (σ, m) be such that $m_{\sigma(i)} = x_{\sigma(i)} + d$ for

all $i \in Q$. Then, by quasi-linearity and monotonicity of preferences in money,

$$(\sigma(i), m_{\sigma(i)}) I_i(\emptyset, v_{i\sigma(i)} + m_{\sigma(i)}) R_i(\emptyset, v_{i\sigma(j)} + m_{\sigma(j)}) I_i(\sigma(j), m_{\sigma(j)})$$

Thus, $(\sigma, m) \in N(e)$.

We have just proved that for any assignment with the above characteristics, there exists an *envy-free* allocation. Taking an efficient assignment an allocation exists that is *envy-free* and, by Lemma 2.1, is *Pareto efficient*. ■

Proof of Proposition 2.3. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ be such that $M \geq qv_{iA} - \sum_{B \in \mathcal{P}} v_{iB}$ for all $i \in Q$, for all $A, B \in \mathcal{P}$, for all $\mathcal{P} \in \mathcal{P}(\Omega, q)$. To

prove that an envy-free allocation exists with $m \geq 0$ it is enough to see that, under the condition imposed on M , the allocation obtained in the above proposition is such that $m \geq 0$. For each $i \in Q$,

$$m_{\sigma(i)} = x_{\sigma(i)} + d = x_{\sigma(i)} + \frac{M - \sum_{j \in Q} x_{\sigma(j)}}{q}$$

$$m_{\sigma(i)} \geq \frac{q(x_{\sigma(i)} + v_{i\sigma(i)}) - (\sum_{j \in Q} x_{\sigma(j)} + \sum_{j \in Q} v_{i\sigma(j)})}{q}$$

Since $v_{i\sigma(i)} + x_{\sigma(i)} \geq v_{i\sigma(j)} + x_{\sigma(j)}$ for all $j \in Q$,

$$q(x_{\sigma(i)} + v_{i\sigma(i)}) \geq (\sum_{j \in Q} x_{\sigma(j)} + \sum_{j \in Q} v_{i\sigma(j)})$$

Therefore, $m_{\sigma(i)} \geq 0$ for all $i \in Q$.

let $z = (\sigma, m) \in N(e)$ be given. Then, it is easy to see that for all $i \in Q$,

$$v_{i\sigma(i)} + m_{\sigma(i)} \geq \frac{\sum_{B \in \mathcal{P}} v_{iB} + M}{q} \quad \text{where } \mathcal{P} = \sigma(Q)$$

Thus, by the condition on M , $\frac{\sum_{B \in \mathcal{P}} v_{iB} + M}{q} - v_{i\sigma(i)} \geq 0$. Therefore,

$m_{\sigma(i)} \geq 0$ for all $i \in Q$. ■

Proof of Lemma 3.1. Let $e = (Q, \Omega, M; R_Q) \in \mathcal{E}_{q1}$ be such that $Q = \{1, 2\}$, and let $Q' = \{1, 2, 3\}$, let $\Omega' = \Omega \cup \{\alpha_o\}$. For each $\mathcal{A} \in \mathcal{P}(\Omega)$, and for each $i \in Q$, let $m_{\mathcal{A}}^i \in \mathbb{R}$ be such that $z_i I_i(\mathcal{A}, m_{\mathcal{A}}^i)$. Let $m_o = \max_{i, \mathcal{A}} \{m_{\mathcal{A}}^i\}$, and let $\bar{M} > m_o$, and let $M' = M + m_o$. Let R'_Q be such that

(i) for all $i \in Q$, $R'_i|_{P(\Omega) \times \mathbb{R}} = R_i$, $z_i I'_i(\alpha_o, m)$ and $z_i I'_i(\mathcal{A}, \bar{M})$ for all $\mathcal{A} \in P(\Omega')$ such that $\alpha_o \in \mathcal{A}$ and $\mathcal{A} \neq \{\alpha_o\}$,

(ii) for all $i \in Q$, $(\alpha_o, m_o)I'_3(\sigma(i), m_{\sigma(i)})$ and $(\alpha_o, m_o)I'_3(\mathcal{A}, \bar{M})$ for all $\mathcal{A} \in P(\Omega')$ such that $\mathcal{A} \neq \{\alpha_o\}$, $\mathcal{A} \neq \sigma(i)$.

Let $z' = (\sigma', m')$ be such that $\sigma'_o = \sigma$, $\sigma'(3) = \alpha_o$, $m'_o = m$, and $m'_3 = m_o$. Clearly, $e = r^{z'}_o(e')$, $z'_o = z$ and z' is *envy-free* for the economy e' . We still have to prove that z' is *Pareto efficient* for the economy e' , and that for all $z'' \in NP_{\varphi}(e')$, $z'' \approx z'$.

Step 1. $z' \in P(e')$

Suppose that z' is not *Pareto efficient* for e' . Then, by Lemma 2.1, σ' is not an efficient assignment for e' . An assignment τ exists such that

$$\sum_{i \in Q'} v_{i\tau(i)} > \sum_{i \in Q'} v_{i\sigma'(i)} \quad [1]$$

Claim 1. The assignment τ is not a permutation of σ' .

Suppose that a permutation $\pi: Q \rightarrow Q$ exists such that $\tau(i) = \sigma'(\pi(i))$. Since z' is *envy-free* for e' , $v_{i\sigma'(i)} + m'_{\sigma'(i)} \geq v_{i\sigma'(j)} + m'_{\sigma'(j)}$ for all $i, j \in Q'$. Thus,

$$\begin{aligned} v_{i\sigma'(i)} + m'_{\sigma'(i)} &\geq v_{i\tau(i)} + m'_{\tau(i)} \quad \text{for all } i \in Q' \\ v_{i\sigma'(i)} &\geq v_{i\tau(i)} + m'_{\tau(i)} - m'_{\sigma'(i)} \quad \text{for all } i \in Q' \\ \sum_{i \in Q'} v_{i\tau(i)} &\leq \sum_{i \in Q'} v_{i\sigma'(i)} \quad \text{in contradiction to [1]} \end{aligned}$$

Claim 2. The assignment τ does not satisfy $\bigcup_{i \in Q} \tau(i) = \Omega$.

Suppose that $\bigcup_{i \in Q} \tau(i) = \Omega$. Then, condition [1] implies that

$$\sum_{i \in Q'} v_{i\tau(i)} > \sum_{i \in Q'} v_{i\sigma'(i)} = \sum_{i \in Q'} v_{i\sigma(i)}$$

in contradiction to the *Pareto efficiency* of z for the economy e .

Claim 3. In the assignment τ no agent in Q receives $\{\alpha_o\}$.

Suppose, without loss of generality, that $\tau(1) = \{\alpha_o\}$. By Claim 1, we know that τ is not a permutation of σ' , thus, $\tau(3) = \mathcal{A}$ such that $\mathcal{A} \neq \{\alpha_o\}$, $\mathcal{A} \neq \sigma(i)$ for all $i \in Q$, and $\tau(2) = \mathcal{B}$, with $\mathcal{B} \in P(\Omega)$. Thus,

$$\begin{aligned} v_{1\tau(1)} &= v_{1\sigma(1)} + m_{\sigma(1)} - m_o \\ v_{2\tau(2)} &= v_{2\sigma(2)} + m_{\sigma(2)} - m_{\mathcal{B}}^2 \\ v_{3\tau(3)} &= v_{3\sigma'(3)} + m_o - \bar{M} \end{aligned}$$

For \bar{M} sufficiently large, $\sum_{i \in Q} v_{i\tau(i)} \leq \sum_{i \in Q} v_{i\sigma'(i)}$ in contradiction to [1].

Claim 4. In the assignment τ , $\tau(3) \neq \sigma(i)$ for all $i \in Q$.

Suppose that $\tau(3) = \sigma(1)$. By Claim 1, we know that τ is not a permutation of σ' , thus, for some agent in Q , $\tau(i) = \mathcal{A}$ such that $\mathcal{A} \neq \{\alpha_o\}$, and $\alpha_o \in \mathcal{A}$, without loss of generality suppose that $i = 1$, and $\tau(2) = \mathcal{B}$, with $\mathcal{B} \in P(\Omega)$. Thus,

$$\begin{aligned} v_{1\tau(1)} &= v_{1\sigma(1)} + m_{\sigma(1)} - \bar{M} \\ v_{2\tau(2)} &= v_{2\sigma(2)} + m_{\sigma(2)} - m_{\mathcal{B}}^2 \\ v_{3\tau(3)} &= v_{3\sigma'(3)} + m_o - m_{\sigma(1)} \end{aligned}$$

For \bar{M} sufficiently large, $\sum_{i \in Q} v_{i\tau(i)} \leq \sum_{i \in Q} v_{i\sigma'(i)}$ in contradiction to [1].

Claim 5. In the assignment τ , $\tau(3) \neq \mathcal{A}$, with $\mathcal{A} \in \mathcal{P}(\Omega')$, $\alpha_o \in \mathcal{A}$, $\mathcal{A} \neq \{\alpha_o\}$.

The proof of this claim is similar to the one in Claim 4. We omit it.

By Claim 1 to 5, the assignment τ does not exist, therefore, the allocation z' is *Pareto efficient* in the economy e' .

Step 2. For all $z'' \in \text{NP}_{\mathcal{P},(e')}$, $z'' \approx z'$.

Suppose, without loss of generality, that $\mathcal{P}' = \{\mathcal{A}, \mathcal{B}, \{\alpha_o\}\}$. Let $z'' = (\tau, m'') \in \text{NP}_{\mathcal{P},(e')}$, then,

Claim 1. $m''_{\alpha_o} \leq m'_{\alpha_o}$.

Suppose that $m''_{\alpha_o} > m'_{\alpha_o}$. Since $z'' \in \text{NP}_{\mathcal{P},(e')}$, for all $i \in Q'$, $z''_i R'_i(\alpha_o, m''_{\alpha_o}) P'_i(\alpha_o, m'_o) I'_i z'_i$, in contradiction to the *Pareto efficiency* of z' for e' .

Claim 2. Let $i \in Q'$ be such that $\sigma(i) = \mathcal{A}$, $\tau(i) = \{\alpha_o\}$. Then, $m''_{\mathcal{A}} \leq m'_{\mathcal{A}}$.

Since $z', z'' \in \text{NP}_{\mathcal{P},(e')}$, $(\mathcal{A}, m'_{\mathcal{A}}) I_i(\alpha_o, m'_{\alpha_o}) R_i(\alpha_o, m''_{\alpha_o}) R_i(\mathcal{A}, m''_{\mathcal{A}})$. By monotonicity of preferences in money, $m'_{\mathcal{A}} \geq m''_{\mathcal{A}}$.

Claim 3. Let $j \in Q$ be such that $\tau(j) = \mathcal{A}$. Then, $m''_{\mathcal{B}} \leq m'_{\mathcal{B}}$.

Since $z', z'' \in \text{NP}_{\mathcal{P},(e')}$, $(\mathcal{B}, m'_{\mathcal{B}}) R_j(\mathcal{A}, m'_{\mathcal{A}}) R_j(\mathcal{A}, m''_{\mathcal{A}}) R_j(\mathcal{B}, m''_{\mathcal{B}})$. By monotonicity of preferences in money, $m'_{\mathcal{B}} \geq m''_{\mathcal{B}}$.

By Claim 1,2 and 3, $m'_{\mathcal{A}} = m''_{\mathcal{A}}$, $m'_{\mathcal{B}} = m''_{\mathcal{B}}$, $m'_{\alpha_o} = m''_{\alpha_o}$. Furthermore, since z' and z'' are envy free, $z''_i I'_i z'_i$ for all $i \in Q$. Thus, $z'' \approx z'$. ■

Step 1 in Proposition 4.2 . Under substitutability, for all $S \subseteq Q$, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \Omega$ such that $\mathcal{A} \subseteq \mathcal{B}$

$$U_S(\mathcal{B} \cup \mathcal{C}) - U_S(\mathcal{B}) \leq U_S(\mathcal{A} \cup \mathcal{C}) - U_S(\mathcal{A})$$

Proof. We proceed by induction on the cardinality of \mathcal{C} . Let $\mathcal{C} = \{\beta\}$, $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{B} \setminus \mathcal{A} = \{\alpha_1, \dots, \alpha_k\}$. Then, applying substitutability,

$$\begin{aligned}
U_S(\mathcal{B} \cup \mathcal{C}) - U_S(\mathcal{B}) &= U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-1}\} \cup \{\alpha_k, \beta\}) - U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-1}, \alpha_k\}) \leq \\
&\leq U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-1}\} \cup \{\beta\}) - U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-1}\}) \leq \\
&\leq U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-2}\} \cup \{\beta\}) - U_S(\mathcal{A} \cup \{\alpha_1, \dots, \alpha_{k-2}\}) \leq \dots \leq \\
&\leq U_S(\mathcal{A} \cup \{\beta\}) - U_S(\mathcal{A}) = U_S(\mathcal{A} \cup \mathcal{C}) - U_S(\mathcal{A}).
\end{aligned}$$

By hypothesis of induction for $|\mathcal{C}| = k-1$

$$U_S(\mathcal{B} \cup \mathcal{C}) - U_S(\mathcal{B}) \leq U_S(\mathcal{A} \cup \mathcal{C}) - U_S(\mathcal{A})$$

Let \mathcal{C} be such that $|\mathcal{C}| = k$, and let $\beta \in \mathcal{C}$, let $\mathcal{B}' = \mathcal{B} \cup \{\beta\}$, $\mathcal{A}' = \mathcal{A} \cup \{\beta\}$ and $\mathcal{C}' = \mathcal{C} \setminus \{\beta\}$ then, applying induction hypothesis,

$$\begin{aligned}
U_S(\mathcal{B} \cup \mathcal{C}) - U_S(\mathcal{B}) &= U_S(\mathcal{B}' \cup \mathcal{C}') - U_S(\mathcal{B}') + U_S(\mathcal{B}') - U_S(\mathcal{B}) \leq U_S(\mathcal{A}' \cup \mathcal{C}') - U_S(\mathcal{A}') + \\
&+ U_S(\mathcal{A}') - U_S(\mathcal{A}) = U_S(\mathcal{A} \cup \mathcal{C}) - U_S(\mathcal{A}). \blacksquare
\end{aligned}$$

R E F E R E N C E S

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