LAGRANGEAN CONDITIONS FOR GENERAL OPTIMIZATION PROBLEMS WITH APPLICATIONS TO CONSUMER THEORY[1]

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WP-AD 90-02

[1] Thanks are due to L. Corchón, I. Ortuño and A. Villar for helpful comments.The second author thanks the D.G.I.C.Y.T. for the financial support of projects PB86 - 289 and PS89 -0066.

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Editor: Instituto Valenciano de
Investigaciones Económicas, S.A.
Primera Edición Diciembre 1990
ISBN: 84-7890-312-7
Depósito Legal: V-3245-1990
Impreso por KEY, S.A., Valencia
Cardenal Benlloch, 69, 46021-Valencia.

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ABSTRACT

In this paper we provide Lagrangean conditions in a problem general enough to encompass both the classical quasiconcave optimization problem and the maximization of binary relations.

The formal results are applied to consumer theory, providing analytical characterizations of the solution to the consumer problem in some cases where preferences are not representable by continuous utility functions.

1. INTRODUCTION.

Consider the following description of a general optimization problem: Let C denote a reference space, and let $\mathbb{P}$ stand for a binary relation on C, such that, for each $x,y \in C$, "$x\mathbb{P}y$" means that point x "precedes", "is greater than" or "is preferred to" point y. For a given subset $A \subset C$, to be interpreted as the set of feasible alternatives, find a point $x^* \in A$ such that:

$$\{ x \in A \mid x \mathbb{P} x^* \} = \emptyset$$

Concerning this optimization problem, there are two related but substantially different questions which immediately arise:

(a) Does it have a solution?

(b) What does it look like?

To deal with the first question implies finding sufficent conditions to ensure that an optimum exists whenever the feasible set is nonempty. When $\mathbb{P}$ is a transitive and continuous relation , and C is a connected subset of a topological vector space, $\mathbb{P}$ can be represented by a continuous function, $f:C \to \mathbb{R}$, so that $x \mathbb{P} y$ iff $f(x) > f(y)$ [see Debreu (1959, ch.4), Schmeidler (1971)]. In this case, the optimization problem takes the form:

$$\begin{array}{c} \text{Max } f(x) \\ \text{s.t. } x \in A \end{array}$$

The existence of a solution to this problem can then be approached via Weierstrass' Theorem, when A is a nonempty, closed set that is bounded from above.

In many relevant cases, the binary relation is not representable by a continuous function, mainly due to lack of transitivity (as happens when $\mathbb{P}$ refers to individual preferences in economic theory). In these cases, the existence of a solution to the optimization problem can be ensured when the binary relation is continuous and convex, via the application of the Knaster-Kuratowski-Mazurkiewicz Theorem [as analyzed in Sonnenschein (1971) or Shafer (1974)].

One way of answering the second question consists of finding some analytical (or geometrical) properties that an optimum may satisfy. A complete treatment of this point is obtained when such properties are necessary and sufficient conditions, so that they provide us with a suitable characterization of the solutions.

The characterization problem has been dealt with in the classical optimization literature. The main concern is to obtain sufficient conditions to solve the problem

$$\begin{aligned} &\text{Max } f(x) \\ &\text{s.t. } g_i(x) \geq 0 \end{aligned} \qquad \text{(COP)}$$

where f, g_i, $i \in I$ are continuous real valued functions defined over some topological vector space. In the case whereby $\mathbb{P}$ is transitive, continuous and convex, f turns out to be a quasi-concave function, and then sufficient conditions may be obtained by means of results on separability of convex sets. These sufficient conditions (lagrangean-type) turn out to be characterizations of the solutions to (COP) under constraint qualifications [see the classical results by Kuhn & Tucker (1951), Arrow & Enthoven (1961)].

In this paper we analyze the characterization problem in the case where $\mathbb{P}$ is a convex binary relation dispensing with the transitivity hypothesis, so that the optimization problem cannot be formulated as the maximization of a real valued function.

In Section 2 we present the main framework. Section 3 is devoted to the analysis of the consumer problem, as a consequence of our previous results to the case of maximization of binary relations. Our main contribution consists of providing a characterization of the solution to the consumer problem in some cases where preferences are not representable by a continuous utility function. These cases are closely related to those studied in the economics literature [Fan (1961); Sonnenschein (1971); Shafer (1974), Mas-Colell (1974), Gale & Mas-Colell (1975) or Shafer and Sonnenschein (1975)].

2. THE MAIN FRAMEWORK

Let $F[\mathcal{T}]$ be a locally convex (Hausdorff) real space, and $X \subseteq F$. We consider a non-empty family $\{S_i\}_{i\in I}$ of convex sets in X. We write $S = \bigcap_{i\in I} S_i$.

Let us consider now the family Ψ of those set-valued mappings $T:X \rightarrow\rightarrow X$ such that for every $x \in X$, (a) $T(x) \in \mathcal{T}$; (b) $T(x)$ is a convex set, and (c) $x \in cl[T(x)]$ whenever $T(x) \neq \emptyset$.

If $T \in \Psi$, we consider the problem[2]:

$$T(x) \cap S = \emptyset$$
$$\text{s.t. } x \in S \qquad\qquad P(T)$$

Given $x \in X$, we write $\Psi(x) = \{ T \in \Psi\text{: } x \text{ solves } P(T)\}$.

For $M \subseteq F$, we denote by $K(M)$ the cone generated by M. If F' stands for the set of continuous linear functionals on F, then the polar of M is $M^\circ = \{ p \in F' \mid \langle p,x \rangle \leq 0,\ \forall x \in M\}$, and the discriminant of M [cf. Gutierrez (1985)] is $D(M) = \{(p,\alpha) \in F'\times\mathbb{R} \mid \langle p,x\rangle \leq \alpha,\ \forall x \in M\}$. M° is then

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[2] Notice that if X is convex and open and we consider a family $\{g_i\}_{i\in I}$ of quasiconcave functions $g_i:X \rightarrow \mathbb{R}$, and $\Psi' = \{f:X \rightarrow \mathbb{R} \mid f \text{ is continuous and quasiconcave}\}$, the classical quasiconcave optimization problem for $f \in \Psi'$:

$$\text{Max } f(x)$$
$$\text{s.t. } g_i(x) \geq 0 \qquad\qquad P'(f)$$

turns out a particular case of $P(T)$ by taking $S_i=\{z\in X \mid g_i(z)\geq 0\}$, $i\in I$, and $T(x) = \{z \in X \mid h(z) > h(x)\}$.

a (weakly) closed convex cone in **F'**, and **D(M)** a (weakly) closed convex cone in $F'\times\mathbb{R}$. For any set C, $\mathcal{P}(C)$ is the set of all possible subsets of C and $\mathcal{P}_f(C)$ is the set of all finite subsets of C.

In the following definitions, we consider some properties on the family $\{S_i\}_{i\in I}$:

Definition 1.- We shall say that $\{S_i\}_{i\in I}$ satisfies property Ñ iff

$$D\Big(\bigcap_{i\in I} S_i\Big) = co\Big[\bigcup_{i\in I} D(S_i)\Big]$$

Definition 2.- Let $x \in X$. We shall say that $\{S_i\}_{i\in I}$ satisfies property Ñ(x) iff

$$\Big[\bigcap_{i\in I}(S_i-x)\Big]^{\circ} = co\Big[\bigcup_{i\in I}(S_i-x)^{\circ}\Big]$$

Property Ñ is a global property, and property Ñ(x) is the corresponding local property. Property Ñ and property Ñ(x) could have been defined through the inclusions $D(S) \subseteq co\,[\bigcup_{i\in I} D(S_i)]$ and $(S-x)^{\circ} \subseteq co\,[\bigcup_{i\in I}(S_i-x)^{\circ}]$, respectively, since the opposite inclusions always hold.

If the sets S_i are closed cones with vertex at x, then $(S-x)^{\circ} = cl\{co[\bigcup (S_i-x)^{\circ}]\}$ [cf. Köthe (1969)]. On the other hand, if the sets S_i are closed and $S \neq \emptyset$, then $D(S) = cl\{co[\bigcup_{i\in I} D(S_i)]\}$ [cf. Gutierrez (1985)].

Let $s \in S$. If the sets S_i are closed, a sufficient condition for property Ñ(s) to hold is that $[\bigcap_{i\in I} int(S_i)] \neq \emptyset$ [cf. Holmes (1972)]. In the finite dimensional case it is enough that $[\bigcap_{i\in I} \text{rint}(S_i)] \neq \emptyset$, and it is not necessary to require the sets S_i to be closed [cf. Rockafellar (1970)]. In a parallel way, in the finite dimensional case, if $[\bigcap_{i\in I} \text{rint}(S_i)] \neq \emptyset$, then $\tilde{N}$ holds [cf. Gutierrez (1985)].

The following proposition specifies the relationship between $\tilde{N}$ and $\tilde{N}(s)$:

Proposition 1[3]**.-** (a) If property $\tilde{N}$ is satisfied, then $\tilde{N}(s)$ holds $\forall s \in S$.

(b) If S is compact, $\tilde{N}$ holds iff $\tilde{N}(s)$ holds $\forall\ s\in S$.

Proof (a) Suppose $\tilde{N}$ holds, and let $s \in S$. If $p \in (S-s)^{\circ}$, then $(p,p(s)) \in D(S)$, and thus $J \in \mathcal{P}_f(I)$ exists such that $(p,p(s)) = \sum_{j\in J} (p_j,\alpha_j)$, with $(p_j,\alpha_j) \in D(S_j)$. Moreover, $p_j(s) = \alpha_j$ for every $j \in J$: obviously, $\langle p_j,s\rangle \leq \alpha_j$; if, for some $k \in J$, $\langle p_k,s\rangle < \alpha_k$, then $\langle p,s\rangle = \sum_{j\in J} \langle p_j,s\rangle < \sum_{j\in J} \alpha_j = \langle p,s\rangle$, which would be a contradiction. Then, $p_j \in (S_j-s)^{\circ}$. Therefore $p \in \text{co}[\bigcup_{i\in I} (S_i-s)^{\circ}]$, and property $\tilde{N}(s)$ follows.

(b) Suppose now that S is compact, and $\tilde{N}(s)$ holds $\forall\ s \in S$. If $(p,\alpha) \in D(S)$, let $s_0 \in S$ such that $\langle p,s_0\rangle = \sup \{\langle p,s\rangle,\ s \in S\}$. Thus, $\langle p,s_0\rangle \leq \alpha$; applying $\tilde{N}(s_0)$, immediately $(p,\langle p,s_0\rangle) \in \text{co}[\bigcup_{i\in I} D(S_i)]$, and $(p,\alpha) \in \text{co}[\bigcup_{i\in I} D(S_i)]$. Hence property $\tilde{N}$ is satisfied. ▮

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[3] Results in section 2 appear in Gutierrez (1989).

Let us now turn to problem P(T). If $T(x) = \emptyset$, then x obviously solves P(T), whatever $\{S_i\}_{i\in I}$ is (unconstrained solution). Let $s \in S$, and consider the following condition[4]:

If $T(s) \neq \emptyset$, then: $\exists\ p\in[(T(s)-s)^{\circ}-\{0\}],\ \exists\ J\in\mathcal{P}_{f}(I),\ \exists\ p_j\in(S_j-s)^{\circ}, j\in J\ /\ p+\sum_{j\in J} p_j=0$	L(T,s)

Then, we get the following result:

Proposition 2.- Let $s \in S$. Then:

(a) $L(T,s) \Rightarrow s$ solves P(T)

(b) Under $\tilde{N}(s)$: s solves $P(T) \Rightarrow L(T,s)$.

Proof: (a) If $s_0 \in (T(s)\cap S)$ exists, then $\langle p + \sum_{j\in J} p_j, s_0-s\rangle < 0$ (since $\langle p,s_0\rangle < \langle p,s\rangle$, as T(s) is open), and so L(T,s) would be contradicted.

(b) If s solves P(T), then (applying the separation theorem and taking into account that $s \in [cl(T(s)) \cap S])$, $p \in F'$, $p \neq 0$, exists such that $p \in (T(s)-s)^{\circ}$, and $-p \in (S-s)^{\circ}$, and thus, L(T,s) follows from $\tilde{N}(s)$. ∎

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[4] L(T,s) is the announced general lagrangean condition. Notice that, in applying it to the quasiconcave case, we get:

$\exists\xi \in df(s)\ \exists J\in\mathcal{P}_f(I(s))\ \exists\xi_j\in dg_j(s)\ \exists\lambda_j\geq 0,\ j\in J\ /\ \xi + \sum \lambda_j\xi_j = 0$, where d stands for the subdifferential , and $I(s) = \{i\in I; g_i(s)=0\}$, the usual lagrangean condition.

Let us consider now the following definitions:

Definition 3.- Let $s \in S$. We say that s is a *regular* point if $L(T,s)$ holds $\forall\ T \in \Psi(s)$.

Definition 4.- A *constraint qualification* for s (CQ-s) is a sufficient condition imposed on $\{S_i\}_{i\in I}$ guaranteeing that s is a regular point.

Definition 5.- A *weakest constraint qualification* for s (WCQ-s) is a (CQ-s) which holds iff s is a regular point.

Then we get the next result:

Proposition 3.- Let $s \in S$. Then property $\tilde{N}(s)$ is a WCQ-s.

Proof: We have seen above that $\tilde{N}(s)$ is a CQ-s. Suppose now that s is a regular point. Let $q \in (S-s)^{\circ}$, $q \neq 0$. We define $T^*:X \to \mathcal{T}$ by $T^*(x) = x + \{z \in F \colon \langle q,z\rangle > 0\}$. Then $T^* \in \Psi(s)$, and $L(T^*,s)$ is satisfied. As $(T(s)-s)^{\circ} = \{-\lambda q,\ \lambda \geq 0\}$, we have that $q \in co[\bigcup_{i\in I} (S_i-s)^{\circ}]$, and property $\tilde{N}(s)$ follows. ∎

Notice that condition $L(T,s)$ is useful in order to obtain points solving $P(T)$. Therefore, it is relevant to know if the family $\{S_i\}_{i\in I}$ satisfies a constraint qualification for every point $s \in S$.

Definition 6.- A *general constraint qualification* (GCQ) is a condition on $\{S_i\}_{i\in I}$ guaranteeing that every point of **S** is regular.

Definition 7.- A *weakest general constraint qualification* (WGCQ) is a GCQ which holds iff every point of **S** is regular.

From definitions 6 and 7 and propositions 1 and 3, we obtain the following corollary:

Corollary 1.- Property $\tilde{N}$ is a GCQ. If **S** is compact, then property $\tilde{N}$ is a WGCQ.

3. AN APPLICATION TO CONSUMER THEORY

Consider now a choice problem in which a consumer has to select one of the best elements in a choice set consisting of consumption bundles (to be referred to as the *consumption set*). The consumer's choice possibilities are actually restricted to a subset of her consumption set, determined by her available wealth.

Let $X \subset \mathbb{R}^n$ denote the consumption set, and let $\mathbb{P}$ denote a binary relation defined on X; where, for each $x,y \in X$, $y \,\mathbb{P}\, x$ is interpreted as "option y is strictly preferred to option x". We shall refer to $\mathbb{P}$ as the consumer's preference relation[5].

A point $\omega \in \mathbb{R}$ stands for the consumer's wealth, and a point π in $\mathbb{R}^n_+$ denotes a price vector. Consider now the following set:

$$W = \{x \in \mathbb{R}^n;\ \langle\pi,x\rangle \leq \omega \}$$

Then, the consumer's feasible set (or *budget set*), at prices π and for a wealth of ω, will be given by $\mathbf{B} = X \cap W$.

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[5] We start by considering the strict preference relation instead of the weak preference relation. This involves no loss of generality, as a strict preference may be converted into a complete weak preference relation by making any noncomparable elements indifferent. This creates no problems as long as we do not require indifference to be transitive.

For each $x \in X$, let us define $U(x) = \{ z \in X : z \mathbb{P} x\}$[6], and let $I = \{1,2\}$, with $S_1 = X$, $S_2 = W$. Then, the consumer's choice problem turns out to be a particular case of P(T) in Section 2, which can be formulated as follows: Find $x^* \in \mathbb{R}^n$ such that,

$$U(x) \cap \mathbf{B} = \emptyset$$

$$\text{s.t. } x \in X \cap W \qquad\qquad P(U)$$

Notice that x solves P(U) if and only if it is a maximal element of $\mathbb{P}$ on **B**.

Suppose now that, for every $x \in X$, (a) U(x) is open ; (b) U(x) is convex, and (c) where $U(x) \neq \emptyset$, then $x \in cl[U(x)]$. Then, the results in Section 2 apply.

Let $b \in \mathbf{B}$. We consider the condition:

$$(\mathbf{B}-b)^\circ \subseteq co[(\mathbf{C}-b)^\circ \cup (W-b)^\circ] \qquad\qquad \tilde{N}'(b)$$

Condition $\tilde{N}'(b)$ is always satisfied if **C** is a polyhedral convex set[7] (discard nonactive linear inequality constraints and apply Farkas Lemma).

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[6] U(x) is sometimes called the *upper contour set* of x. So, we can look at the preference relation as a correspondence U:X->->X.

[7] A polyhedral convex set is a set which can be expressed as the intersection of some finite collection of closed half-spaces [see Rockafellar (1970)].

Then, the results in Section 2 applied to this particular case can be summarized in the following proposition:

Proposition 4.- Suppose U:X ->-> X represents a binary relation such that $x \in X$, (a) U(x) is open (b) U(x) is convex, and (c) if $U(x) \neq \emptyset$, then $x \in cl[U(x)]$. If C is a polyhedral convex set , $b \in B$ solves P'(U) if and only if the following condition holds:

If $U(b) \neq \emptyset$,

then $\exists\ \gamma \in [(U(b)-b)^{\circ}-\{0\}],\ \exists\ k \in (C-b)^{\circ} /\ \gamma+k+\pi=0$ $\qquad L(U,b)$

Consider now the following definitions:

Definition 8.- We say that U is *weakly convex* if U(x) is convex $\forall\ x \in X$.

Definition 9.- We say that U is *convex* if $x \in U(y) \Rightarrow [\lambda x+(1-\lambda)y] \in U(y)$, $\forall\ \lambda \in (0,1)$.

Definition 10.- We say that U is *continuous*[8] if, $\forall\ x \in X$, both U(x) and $U^{-}(x)$[9] are open.

If U is convex and U(x) is open, then U is weakly convex; if U is convex, then condition (c) holds [cf. Debreu,,(1959)]. Moreover, in the

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[8] We consider here the traditional definition of continuous preferences. Notice that this concept coincides with that of having open sections, if we look at U as a set-valued mapping [cf. Border (1985)].

[9] $U^{-}(x) = \{\ y \in X;\ x \in U(y)\ \}$

case whereby U(x) is open and convex, if condition (c) holds, then U is convex [cf. accesibility lemma, Köthe (1969)].

The previous remark allows us to specify Proposition 4 in a different way:

Proposition 4'.- Suppose $U:X \rightarrow\!\!\!\rightarrow X$ represents a convex binary relation such that U(x) is open $\forall\ x \in X$. If **C** is a polyhedral convex set, $b \in \mathbf{B}$ solves P'(U) if and only if the following condition holds:

If $U(b) \neq \emptyset$,

then $\exists\ \gamma \in [(U(b)-b)^{\circ}-\{0\}],\ \exists\ k \in (C-b)^{\circ} /\ \gamma+k+\pi=0$ L(U,b)

Proposition 4' provides a characterization of the individual demand set whenever U(x) is open $\forall\ x \in X$, and U is convex. Notice that, since previous conditions have nothing to do with transitivity, our characterization covers several cases in which preferences are not representable by continuous utility functions.

Then, the following corollaries are obtained:

Corollary 2.- Let $U:X \rightarrow\!\!\!\rightarrow X$ be a continuous and strongly convex[10] binary relation [cf. Shafer (1974)].Then, $b \in \mathbf{B}$ solves P'(U) if and only if L(U,b) holds.

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[10] We shall say that U is strongly convex if $y \notin U^{-}(x)$, $y \notin U^{-}(z) \Rightarrow y \in U[\lambda x+(1-\lambda)z]$, $\forall\ \lambda \in (0,1)$. If U is strongly convex and U(x) is open for every $x \in X$, then U is convex (cf. Debreu, 1959).

Corollary 3.- Let $U:X \to\to X$ be a continuous and convex binary relation. Then, $b \in B$ solves P'(U) iff L(U,b) holds.

Remark: Notice that, under the hypothesis in Corollary 3, U has an open graph [see Shafer (1974) and Bergstrom, Parks & Rader (1976)]. So, Corollary 3 covers those preferences studied by Fan (1961) [cf. Border,7.5 (1985)]. See, as well, Mas-Colell (1974) and Gale & Mas-Colell (1975).

Let us now consider a binary relation $U:X \to\to X$ such that U(x) is open $\forall\, x \in X$, but U(x) is not necessarily convex. In such a case, we consider another binary relation $V:X \to\to X$ given by: V(x) = co U(x). Obviously, V(x) is open and convex $\forall\, x \in X$, and U is majorized by V[11]. We can apply Proposition 4' to V, getting a characterization of those elements solving P'(V). Since any solution to P'(V) is also a solution to P'(U), we get the following result:

Corollary 4.- Let $U:X \to\to X$ a continuous binary relation and $b \in B$. If L(V,b) holds, then b solves P'(U).

Remark: Under the hypothesis in Corollary 4, U has open sections, so we are in the hypothesis of Sonnenschein (1971) and Shafer & Sonnenschein (1975). In these cases, the lagrangean condition L(V,b) turns out to be a sufficient condition for b to belong to the individual demand set.

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[11] That is, U is a subrelation of V, since $U(x) \subseteq V(x)$.

4. CONCLUSIONS.

In this paper we provide a characterization of the solution to a class of optimization problems general enough to cover both the classical quasiconcave optimization problem and the maximization of binary relations. We obtain sufficient Lagrangean conditions for the general problem, and also provide a weakest constraint qualification under which the Lagrangean condition also turns out to be necessary.

We apply our formal results to the consumer problem. In the standard case, our constraint qualification always holds. So, the lagrangean condition appears as a characterization of the individual demand set, under continuity and convexity of the preference relation. If we drop the convexity assumption, the lagrangean condition is still a sufficient condition for optimality. Interestingly enough, our characterization covers a number of preference relations which are not representable by continuous utility functions, such as those studied by Fan (1961), Shafer (1974), Mas-Colell (1974) and Gale & Mas-Colell (1975) whereas the sufficiency result covers the cases in Sonnenschein (1971) and Shafer & Sonnenschein (1975).

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