VECTOR MAPPINGS WITH DIAGONAL IMAGES[1]

C. Herrero and A. Villar*

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* C. Herrero and A. Villar: Instituto Valenciano de Investigaciones Económicas and University of Alicante.

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ABSTRACT

A diagonal image may be defined as a point in the image of a given mapping, whose components are all equal. More generally, we shall say that a mapping $F: D \rightarrow \mathbb{R}^n$, for $D \subset \mathbb{R}^n$ has a quasi-diagonal image at $x \in D$, if $F_i(x) > F_j(x) \Longrightarrow x_i = 0$, for all $i = 1, 2, ..., n$. This paper investigates sufficient conditions for a set-valued mapping to have quasi-diagonal images (in an extended sense). More specifically, we shall show that an upper-hemicontinuous correspondence, with nonempty, compact and convex values, applying the cartesian product of an arbitrary number of simplexes on the corresponding space, has a quasi-diagonal image. Two applications are provided in order to illustrate the usefulness of these points in economic modelling.

I.- INTRODUCTION.

The purpose of this paper is to analyze the existence of a particular type of values in the image space of a standard class of operators defined on arbitrary products of Euclidean spaces. These particular values will be called *diagonal images*, and will be precisely defined later on. Intuitively, a diagonal image is a point in the range of a given mapping, whose components are all equal.

In order to motivate the problem we are going to deal with, let $F:\mathbb{R}^n_+ \to \mathbb{R}^n$ be a single-valued mapping (where $\mathbb{R}^n$ stands for the n-vector space of real numbers, and $\mathbb{R}^n_+ \equiv \{ \mathbf{x} \in \mathbb{R}^n \ / \ \mathbf{x} \geq 0 \}$), and let $\mathbf{e}$ be a point in $\mathbb{R}^n_+$ whose components are all equal to one. Finding a diagonal image may be described as the search for a pair of points $(\mathbf{x}', \lambda)$, with $\mathbf{x}' \in \mathbb{R}^n_+$ and $\lambda \in \mathbb{R}$, such that:

$$F(\mathbf{x}') = \lambda \ \mathbf{e} \qquad [1]$$

When there exists such a pair of points, we shall say that F has a diagonal image at $\mathbf{x}'$ (observe that when $\lambda = 0$, a diagonal image yields a solution to the equation system $F(\mathbf{x}) = 0$).

Suppose now F applies $\mathbb{R}^n_+ \times \mathbb{R}^k_+$ into $\mathbb{R}^{n+k}$. We can also say that F has a diagonal image if a point $\mathbf{x}'$ and two scalars, λ, δ exist such that:

$$\left. \begin{aligned} F_i(\mathbf{x}') &= \lambda \quad , \text{ for all } i = 1,2, \ldots, n \\ F_i(\mathbf{x}') &= \delta \quad , \text{ for } i = n+1, n+2, \ldots, n+k \end{aligned} \right\} \qquad [1']$$

(in general we may have a mapping defined on the cartesian product of an arbitrary number of Euclidean spaces, i.e., given a set of indices I, consider, for any $i \in I$, a natural number n(i), and the euclidean space $\mathbb{R}^{n(i)}$. Let $E = \prod_{i \in I} R^{n(i)}$, and consider a mapping from E into itself. Then, we will require λ_i, $i \in I$ scalars for the definition of a diagonal image).

Consider now the following variant of problem [1]: Find $x^* \in \mathbb{R}^n_+$, and a scalar λ such that, for all j = 1,2, ..., n:

$$\left.\begin{array}{l} F_j(x^*) \geq \lambda \\ F_j(x^*) > \lambda \implies x^*_j = 0 \end{array}\right\} \quad [2]$$

Problem [2] constitutes a generalization of problem [1]. When such a pair of points (x^*, λ) exists, we shall say that F has a *quasi-diagonal image* at x^*. Observe that such a point yields a solution to the complementarity problem:

$$G(\mathbf{x}) \geq 0$$

$$\mathbf{x}\, G(\mathbf{x}) = 0$$

where $G(\mathbf{x}) \equiv F(\mathbf{x}) - \lambda\, \mathbf{e}$, and $\mathbf{x}\, G(\mathbf{x})$ stands for the usual inner product.

Suppose now F applies $\mathbb{R}^n_+ \times \mathbb{R}^k_+$ into $\mathbb{R}^{n+k}$. As in the previous case, we can also say that F has a quasi-diagonal image if a point x^* and two scalars, λ, δ exist, such that:

$$\left.\begin{array}{l} \forall\ j = 1,2,\dots,\ n,\ \ F_j(x^*) \geq \lambda\ \ \&\ \ F_j(x^*) > \lambda \implies x_j^* = 0 \\ \forall\ j = n+1,\dots,\ n+k,\ F_j(x^*) \geq \delta\ \ \&\ \ F_j(x^*) > \delta \implies x_j^* = 0 \end{array}\right\} [2']$$

(similarly, any number of n-vector spaces can be considered).

Problem [2'] constitutes the extension of problem [1'] analogous to [2] with respect to problem [1]. It is clear that all points satisfying [1] (resp. [1']) also satisfy [2] (resp. [2']). Although the converse is not generally true, when x^* is strictly positive, a quasi-diagonal image turns out a diagonal one. As the reader may well guess, quasi-diagonal images can be shown to exist in many contexts where diagonal images are not possible (as it is the case for the function $F:\mathbb{R}^2_+ \to \mathbb{R}^2$ given by: $F_1(x) = x_1$, $F_2(x) = x_1 + x_2 + 2$).

From a formal viewpoint, quasi-diagonal images are closely related to the existence of fixed points and the solvability of variational inequalities and complementarity problems [see Villar (1990) and the references provided there]. Furthermore, the solvability of some economic problems can be formulated as the search for a quasi-diagonal image [see for instance Herrero & Villar (1990)].

Section II analyzes the existence of quasi-diagonal images for set-valued mappings. The key result of this Section shows that an upper-hemicontinuous correspondence, with nonempty, compact and convex values, applying the cartesian product of an arbitrary number of simplexes on the corresponding space, has a quasi-diagonal image.

Sections III and IV provide two different applications, in order to illustrate the usefulness of this result in economic modelling. The first one considers the problem of distributing a bundle of k goods among n agents in the presence of consumption externalities, so that the resulting allocation could be deemed *egalitarian*. Then, the existence of Lindahl equilibria is analyzed in a model with several public goods, when personalized prices may enter the utility functions.

II.- THE EXISTENCE OF QUASI-DIAGONAL IMAGES.

In order to facilitate the exposition, let us introduce the following definition:

Definition.- Let E be a locally convex Hausdorff topological vector space. We shall say that $\Gamma: D \subset E \dashrightarrow E$ is a **regular mapping** on D, if there exists an upper hemicontinuous correspondence, $\mu: D \dashrightarrow \mathbb{R}^n$ such that, for all $x \in D$,

(a) $\mu(x) \subsetneq \Gamma(x)$

(b) $\mu(x)$ is nonempty, compact and convex.

Remark.- A regular corresponce is a (possibly set-valued) mapping containing an upper hemicontinuous subcorrespondence with nonempty, compact and convex values. A particular family of regular mappings is given by those correspondences which allow for continuous selections (this can be ensured if either Γ is nonempty, convex valued and $\Gamma^{-1}(y)$ is open for each y in $\Gamma(D)$ [Browder (1968)], or Γ is lower hemicontinuous, with nonempty, closed and convex values [Michael (1956)]). Trivially, an upper hemicontinuous correspondence with nonempty, compact and convex values is regular, and a single-valued mapping F is regular if and only if it is continuous on D.

Let n(i) denote a positive integer, for any $i \in I$, where I stands for an arbitrary set of indices. Call $E_i = \mathbb{R}^{n(i)}$, and let $E = \prod_{i \in I} E_i$, and consider the following definition:

Definition.- Let $\Gamma: E \dashrightarrow E$ be a given correspondence. We shall say that Γ has a **quasi-diagonal image** at x^* if there is some $y^* \in \Gamma(x^*)$ and scalars λ_i, $i \in I$, such that, $\forall\ i \in I$,

$$y^*_{ij} \geq \lambda_i \quad \& \quad x^*_{ij} = 0 \text{ whenever } y^*_{ij} > \lambda_i \ , \ j = 1,2,\ldots, n(i)$$

Define now for each $i \in I$ the unitary simplex $S_i \subset E_i$:

$$S_i = \{ x \in E_i \ / \ \sum_{j=1}^{n(i)} x_j = 1, \ x_j \geq 0 \}$$

Denote by $\mathcal{S}$ the cartesian product of those simplices, that is,

$$\mathcal{S} \equiv \prod_{i \in I} S_i$$

The following Lemmas will facilitate the proof of our main result:

Lemma 1.- Let X be a topological space, Y_i a compact space, $\forall\ i \in I$, and T_i: X --> Y_i a regular mapping. Let $Y = \prod_{i \in I} Y_i$ and T: X --> Y such that $T(x) = \prod_{i \in I} T_i(x)$. Then, T is regular.

Proof.-

By Tychonoff's Theorem [see Köthe (1969), p. 18], Y and T(x) are compact spaces, for every $x \in X$. Moreover, since Gr T_i is closed $\forall\ i \in I$, Gr T is closed as well [see Border (1985), prop. 11.25], and, since Y is compact, T is regular.

■

Lemma 2.- Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space, T: K --> K an upper hemicontinuous mapping with nonempty, convex and closed values. Then T has a fixed point.

[A proof of this result can be found in Marchi & Martínez Legaz (1989), cor. 3.3, and Istratescu (1981), cor. 10.3.10].

We are now ready to state and prove the main result of this Section:

Theorem.- Let $\Gamma:\mathcal{S}$ --> E be a regular mapping. Then Γ has a quasi-diagonal image.

Proof.-

E is Hausdorff and locally convex [see Köthe (1969), p. 207], and, by Tychonoff's Theorem, S is a compact subset of E.

$\forall\ i \in I$, let T_i: S --> E_i be the projection of Γ on E_i, and F_i = co $T_i(S)$, the convex hull of $T_i(S)$.F_i is a compact and convex set [see Border (1985, 11.16)]. Let $F = \prod_{i\in I} F_i$. F is a compact and convex subset of E, and $\Gamma = \prod_{i\in I} T_i$.

Now define a correspondence γ_i:F ---> S_i as follows:

$$\gamma_i(z) = \{\ y_i \in S_i \ \ /\ \ y_i\ z_i \leq y'_i\ z_i\ ,\ \forall\ y'_i \in S_i\ \}$$

Clearly γ_i is a nonempty, convex-valued correspondence. Furthermore, γ_i is upper hemicontinuous.Then, γ:F --> S, given by $\gamma(z) = \prod_{i\in I} \gamma_i(z)$ is regular.

Define now a new correspondence, ϕ from $\mathcal{S}$xF into itself as follows:

$$\phi(y, z) = \gamma(z) \times \Gamma(y)$$

By construction, ϕ is an upper-hemicontinuous correspondence with nonempty, compact and convex values, applying a compact and convex set into itself. Thus, Lemma 2 applies, and a point (y^*, z^*) in $\phi(y^*, z^*)$ exists, so that,

$$y^* \in \gamma(z^*), \qquad z^* \in \Gamma(y^*)$$

By definition of γ we have:

$y^*_i\ z^*_i = \min.\ x_i\ z^*_i$, for all x_i in S_i, that is,if we call $\min z^*_i = \lambda_i$, we get

$$z^*_{i1}y^*_{i1} + \ldots + z^*_{in(i)}y^*_{in(i)} \geq \lambda_i$$

since the first part of the inequality is a convex combination of numbers which are equal to or greater than λ_i. Hence the equality is only possible if $z^*_{ij} = \lambda_i$ whenever $y^*_{ij} > 0$, for all $i \in I$, all $j = 1,2,\ldots, n(i)$.

$\blacksquare$

What this Theorem means is that we can find points y^* in $\mathcal{S}$, z^* in $\Gamma(y^*)$ such that, for each $i \in I$, every z^*_{ij} [$j = 1,2,\ldots, n(i)$] either takes on a common value (λ_i, say), or else y^*_{ij} equals zero. Thus in particular, when y^* is an interior solution we have:

$$z^*_{i1} = z^*_{i2} = \ldots . = z^*_{in(i)} = \lambda_i \qquad i \in I$$

Remark.- The same proof can be applied in order to show the existence of points $x^* \in \mathcal{S}$, $y^* \in \Gamma(x^*)$ such that,

$$y^*_{it} < \max_j y^*_{ij} \implies x^*_{it} = 0$$

Let now S^k stand for the cartesian product of k simplexes in $\mathbb{R}^q_+$ (that is, $n(1) = n(2) = \dots = n(k) = q$, and $S^k \subset \mathbb{R}^{qk}_+$). The following Corollary is obtained:

Corollary.- Let $\Gamma: S^k \to \mathbb{R}^n$ be a regular correspondence. Then, there exist points $x^* \in S^k$, $y^* \in \Gamma(x^*)$, such that,

$$y^*_i > y^*_j \implies x^*_{i1} = x^*_{i2} = \dots = x^*_{ik} = 0$$

<u>Proof.-</u>

Define a mapping $\Psi: S^k \to \mathbb{R}^{nk}$ as follows: for each $x \in S^k$

$$\Psi(x) = \Gamma(x) \times \Gamma(x) \times \dots \times \Gamma(x)$$

that is, $\Psi(x)$ stands for the cartesian product of k identical sets, $\Gamma(x)$. Clearly Ψ satisfies the hypothesis of the Theorem and then there exist points $x^* \in S^k$, $p^* \in \Psi(x^*)$ such that

$$p^*_{it} > \beta_i \min_j p^*_{ij} \implies p^*_{ij} = 0$$

By construction,

$$p^* = (y^*, y^*, \dots, y^*)$$

Hence the result follows.

$\blacksquare$

II.- APPLICATIONS (I): THE EXISTENCE OF EGALITARIAN ALLOCATIONS.

Consider a public choice problem in which a planner has to allocate *k* divisible goods among *n* agents by means of transfers, in an environment characterized by the presence of consumption externalities. We shall assume that agents' preferences may be represented by continuous utility functions, and that either interpersonal welfare comparisons can be performed in ordinal terms, or that payoff functions represent relative gains for non-comparable but cardinal utilities [see Villar (1988), Herrero & Villar (1990) for a detailed discussion of this kind of problem].

Let i = 1, 2, ..., n be the subscript identifying the different agents, and j = 1, 2, ..., k that identifying the available goods. A point $x_i \in \mathbb{R}^k_+$, $x_i = (x_{i1}, x_{i2}, \ldots, x_{ik})$, denotes the amounts of goods corresponding to the *ith* agent. A point $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^{nk}_+$ will be called an *allocation*. We shall suppose that all goods are "private goods".

Each agent is assumed to have a continuous utility function defined over entire allocations, $u_i:\mathbb{R}^{nk}_+ \text{-->} \mathbb{R}$. Then, u(x) is the n-vector of utility values for a given allocation, that is,

$$u(x) = [\ u_1(x),\ u_2(x),\ \ldots,\ u_n(x)\ ]$$

where $u:\mathbb{R}^{nk}_+ \text{-->} \mathbb{R}^n$.

An allocation $x^* \in \mathbb{R}^{nk}_+$ will be called *egalitarian* if, for all i, t = 1,2,..., n, $u_i(x^*) > u_t(x^*)$ implies $x^*_i = 0$. That is, we shall say that an allocation is egalitarian when all individuals have the same payoffs, or else those with higher ones receive nothing).

Let $\omega \in \mathbb{R}_+^k$ denote the bundle of commodities to be distributed. Let $S(\omega_j) = \{ z \in \mathbb{R}_+^n \;/\; \sum_{i=1}^{k} z_i = \omega_j \}$. A point in $S(\omega_j)$ describes a distribution of the available amount of commodity j between the n agents. Call $S(\omega) = \prod_{j=1}^{k} S(\omega_j)$. A point in $S(\omega)$ describes an allocation of ω.

The following result is obtained as an immediate application of the Corollary.

Proposition 1.- Let $\omega \in \mathbb{R}_+^k$ be a given bundle of goods, and $u:\mathbb{R}_+^{nk} \to \mathbb{R}^n$ a continuous n-vector of utility functions. Then, there exists an egalitarian allocation $x^* \in S(\omega)$.

This Proposition ensures that we can always distribute a given bundle of commodities so that all individuals getting a positive amount of at least one good, will have the same payoffs. All kind of consumption externalities are allowed for, provided utilities are continuous.

IV.- APPLICATIONS (II): THE EXISTENCE OF LINDAHL EQUILIBRIA IN A MODEL WITH SEVERAL PUBLIC GOODS AND PRICE EXTERNALITIES.

Consider an economy in which there are k public goods and a single private one [see Milleron (1972), for details]. We want to analyze the existence of Lindahl equilibria when prices may affect utilities.

The possibility of prices affecting utilities has been analyzed in private goods economies [see for instance Kalman (1968), Arrow & Hahn (1971, Ch. 6), Grandmont (1983, Ch. 1)]. In the context of public goods this seems a very natural framework: my willingness to pay may depend on what others are going to pay.

There are *n* consumers, each one endowed with ω_j units of the private good (j = 1,2,..., n). $X_j \subset \mathbb{R}_+^{k+1}$ stands for the jth consumer consumption set. Consumers welfare depends on the amounts of public and private goods they enjoy, and on the vector of Lindahl prices. Hence we can write the jth consumer utility function as follows:

$$u_j : X_j \times \mathbb{R}_+^{nk} \dashrightarrow \mathbb{R} \qquad , \quad j = 1,2,\dots, n$$

where $u_j(x_j, y, p)$ makes explicit that the jth consumer welfare depends on the private good she consumes, x_j, the amounts of public goods provided, $y = (y_1, y_2, \dots, y_k)$, and the vector of Lindahl prices,

$$p = (p_{ij}) \qquad , \ i = 1,2,\dots, k \quad ; \quad j = 1,2,\dots, n$$

where p_{ij} denotes the jth consumer contribution to the provision of the ith public good.

Concerning consumers we shall assume:

A.1.- For each $j = 1,2,\ldots, n$:

(a) X_j is a nonempty, compact and convex subset of $\mathbb{R}_+^{k+1}$.

(b) u_j is continuous and quasi-concave in its arguments, and strictly increasing in x_j.

(c) $\omega_j \in \text{rel.int.}X_j$.

This is a standard assumption. In particular, let us point out that the compactness of X_j is assumed for the sake of simplicity, and that the insatiability hypothesis in (b) implies that the private good will always have a positive exchange value (so that it can be taken as the *numéraire*).

For each vector of Lindahl prices, $p \in \mathbb{R}_+^{nk}$, the jth consumer demand is obtained from the solution to the following program:

$$\text{Max. } u_j(x_j, y^j, p)$$

s.t.

$$x_j + \sum_{i=1}^{k} p_{ij}\, y_i^j = \omega_j$$

For each given $p = \bar{p}$, let $\xi_j(\bar{p})$ stand for the set of solutions to this program. Under assumption (A.1), the following properties follow:

(i) $\xi_j(\bar{p})$ is nonempty, compact and convex.

(ii) ξ_j is upper hemicontinuous (the maximum theorem applies here).

Let now $d_j:\mathbb{R}^{nk}_+ \to \mathbb{R}^k_+$, $j = 1,2,\ldots, n$, stand for the projection of the demand mapping ξ_j on the public goods space (that is, $d_j(p)$ is the jth consumer demand for public goods). Since u_j is increasing in x_j, there is no loss of information when the demand for the private good is ignored. Furthermore, d_j preserves properties (i) and (ii) above[1].

We shall suppose that each public good is competitively produced by single-production firms using the private good as an input, under constant returns to scale. The private good is taken as the *numéraire*.

Each firm producing the ith public good faces an output price given by

$$p_i = \sum_{j=1}^{n} p_{ij}$$

For the sake of simplicity we shall choose units so that the supply correspondence for the ith public good, $i = 1,2,\ldots, k$, is only defined if $p_i \leq 1$.

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[1] Since each d_j may be understood as the intersection of two nonempty and closed-valued upper hemicontinuous correspondences, ξ_j and γ_j, where $\gamma_j(p) = X_j \cap \mathbb{R}^k_+$, for all **p** [see Hildenbrand (1974, Prop. 2a, p. 231)].

Summarizing:

A.2.- Public goods are competitively produced by single-production firms. The aggregate supply correspondence for the ith public good, i = 1,2,..., k , is given by:

$$s_i(p_i) = \begin{cases} 0, \text{ when } p_i < 1 \\ [0, +\infty), \text{ when } p_i = 1 \\ \text{undefined, when } p_i > 1 \end{cases}$$

Then, a *Lindahl Equilibrium* with (possibly) positive supply of public goods is a pair (y^*, p^*) , $y^* \in \mathbb{R}^k_+$, $p^* \in \mathbb{R}^{nk}_+$, such that:

(i) For all i = 1,2,..., k, $\sum_{j=1}^{n} p^*_{ij} = 1$.

(ii) For each j = 1,2,..., n, there is some $y^j \in d_j(p^*)$ satisfying the following two requirements:

(ii,a) $y^j_i \leq y^*_i$

(ii,b) $y^j_i < y^*_i$ implies $p^*_{ij} = 0$

for all i = 1,2,..., k.

That is, p^* yields a Lindahl equilibrium if it equalizes the demand for the ith public good, i = 1,2,..., k, for every consumer with a positive personalized price (that is, a positive contribution to the provision of this public good). Notice that the budget constraint ensures the feasibility of such allocation.

The next result follows:

Proposition 2.- Let an economy satisfying assumptions (A.1) and (A.2). Then, there exists a Lindahl equilibrium for this economy.

Proof.-

First notice that we may restrict our search for equilibrium points to the cartesian product of k unit simplexes, $S = \prod_{i=1}^{k} S_i$, where:

$$S_i = \{ p_i \in \mathbb{R}^n_+ \; / \; \sum_{j=1}^{n} p_{ij} = 1 \}$$

Each of these simplexes describes how the total cost of a public good is distributed among the n consumers. Moreover, every demand correspondence, d_j, is upper-hemicontinuous with nonempty, compact and convex values. Then, defining **Γ(p)** as the cartesian product of those d_j**(p)**, j = 1,2,..., n, the result follows from **the** Theorem.

Finally, the Walras Law ensures that the market for the private good will also be in equilibrium.

$\blacksquare$

Proposition 2 shows that a Lindahl equilibrium exists for an economy in which there are *k* public goods and a single private one, when prices may affect utilities, under constant returns to scale.

Remark.- The availability of an existence result compatible with the presence of price externalities, and applicable to any countable number of simplices, becomes specially interesting when we think of the possibility of a sequence of time periods (the allocation of bundles of public goods involving "the future").

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